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EQUILIBRIUM WITH LIMITED-RECOURSE COLLATERALIZED LOANS

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Abstract. We address a general equilibrium model with limited-recourse collateralized loans. Borrowers are burden to constitute physical collateral guarantees, which are repossessed in case of default and delivered to the associated lenders. In addition, lenders may receive payments over collateral values, since debtor's wealth (physical and financial) can be garnished when commitments are not fully honored. The reimbursement of resources is proportional to the size of claims.

Keywords. Collateralized assets; Bankruptcy, Limited-recourse loans; Equilibrium existence.
JEL Classification. D52, D54.

1. Introduction

In seminal papers, Dubey, Geanakoplos, and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007) introduce default and collateralized loans into the general equilibrium model with incomplete markets. They prove that, allowing for default it is always possible to assure equilibrium existence in incomplete markets, even when real assets are available for trade. Indeed, since the financial sector is linked to physical markets through collateral constraints, the scarcity of commodities induce endogenous Radner bounds on short-sales. This avoids discontinuities that may appear on individuals' demands when the rank of return matrices becomes dependent on asset prices and, therefore, equilibrium existence can be proved.

This model of mortgage loans gives rise to a growing theoretical literature. In finite horizon models, Araujo, Orrillo and Páscoa (2000) and Araujo, Fajardo and Páscoa (2005) made extensions to allow for endogenous collateral. Steinert and Torres-Martínez (2007) include CLO markets, where some claims have priority over others to receive resources obtained by the repossession of collateral guarantees. In the infinite horizon context, Araujo, Páscoa, and Torres-Martínez (2002, 2010) prove equilibrium existence without the need to impose transversality conditions, debts constraints or uniform impatient assumptions.[1] Indeed, Ponzi schemes are endogenously avoided by the

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[1]In infinite horizon incomplete markets models where default is not allowed, transversality conditions (or portfolio constraints) jointly with uniform impatient requirements are imposed to avoids Ponzi schemes. See for instance,

scarcity of physical resources used as collateral guarantees. In the context of Markovian economies, the existence of stationary equilibrium in collateralized asset markets was proved by Kubler and Schmedders (2003). Also, Seghir and Torres-Martínez (2008) prove that collateral allows to increase credit opportunities in economies with incomplete demographic participation.

In all these models the only enforcement mechanism in case of default is the seizure of collateral guarantees. Therefore, borrowers make strategic default delivering the minimum between the original promise and the associated collateral value. However, additional payment enforcement mechanisms may appear, for instance, as institutional reactions to credit crisis where collateral guarantees strongly decrease their values. In this context, Páscoa and Seghir (2009) prove that, when defaulters are punished by harsh linear utility penalties, Ponzi schemes opportunities may appear, and equilibrium may cease to exist. Even more, Ferreira and Torres-Martínez (2010) show that, when collateral guarantees are lower, a persistent effectiveness of any payment enforcement could be incompatible with equilibrium. There is also a positive theory of equilibrium existence in collateralized asset markets when utility penalties for default are allowed, as the results of Páscoa and Seghir (2009) or Martins-da-Rocha and Vailakis (2009, 2010).[2]

On the other hand, in the context of general equilibrium models of bankruptcy with unsecured claims, Araujo and Páscoa (2002) propose a two-period incomplete markets model where resources obtained by the payment of loans and the garnishment of wealth are distributed either in proportion to the size of claims or giving priority to smaller claims to receive the whole payment. The former rule of distribution is implemented assuming that a proportion of agents' wealth is protected from expropriation in case of bankruptcy, while the last rule of distribution is implemented making individuals' exemption asymptotically zero as his debt increases. Thus, when the reimbursement is proportional to claims, the level of exemption of rich agents could be substantially larger than the exemption given to poor consumers. In this context, the existence of equilibrium is proved for nominal asset markets. In a related result, Sabarwal (2003) addresses a finite horizon model with numeraire assets where the exemption in case of bankruptcy may be a fixed amount of the wealth. Thus, his result allows poor agents to have a greater proportion of their wealth protected from garnishment. The author analyzes a proportional reimbursement rule and assumes that borrowing is restricted by credit constraints, which may depend on the history of default. Then, it could have two payment enforcements mechanism in case of default: the garnishment of endowment and the restriction of financial participation. In Araujo and Páscoa (2002) and Sabarwal (2003), commodities are perishable and only partial garnishment of physical endowments is allowed.

Kehoe and Levine (1993), Magill and Quinzii (1994, 1996), Hernandez and Santos (1996) and Levine and Zame (1996).

[2]These results are also extensions of seminal works on default and punishment of unsecured debt (see Dubey, Geanakoplos and Shubik (1990, 2005) and Zame (1993)).

In our model we want to include bankruptcy and the garnishment of wealth in a general equilibrium framework with collateralized credit contracts and securitization of debts. Since collateralized loans are securitized into passtrough securities, we can allow markets to garnish the individuals' wealth associated to financial investment positions. Also, we replace credit limits of models with unsecured claims by collateral constraints. Since resources obtained by the seizure of collateral guarantees are delivered to agents that invest in the associated passtrough security, there is no indetermination of the right over physical guarantees, avoiding any risk about the repossession of collateral. Also, our garnishment rules allow for either proportional exemptions or exemptions that protect poor defaulters, reducing the garnishment to a lower percentage of their wealth.

Our economy is stochastic and has two time periods. Commodities may be durable, perishable or may transform into other goods through the time. Debt contracts are limited-recourse loans backed by physical collateral guarantees. These promises are pooled and securitized into passtrough securities. Different to Geanakoplos and Zame (1997, 2002, 2007) or Steinert and Torres-Martínez (2007), we allow for the garnishment of individual wealth in case that some promise is not fully payed. Therefore, when agent's wealth does not cover the total amount of debt, bankruptcy appears.

Since the possibility of wealth loss when debts are not fully payed may induce non-convexities in budget set correspondences, we assume that a continuum of agents can demand commodities, trade debt contracts, and invest in passtrough securities. We allow for different types of garnishment rules, and resources obtained by confiscation are distributed to lenders proportional to their promises. Since in our model the payment of passtrough securities are endogenous, we will concentrate our attention in non-trivial equilibria. That is, those equilibria where asset payments are positive in at least one state of nature. As in Steinert and Torres-Martínez (2007) we can trivially prove the existence of equilibrium when passtrough securities payments are zero, since the economy can be reduced to a pure spot market economy (assuming that debt-contracts have zero price too).

The existence of equilibrium is carried out appealing to the existence of pure strategy Nash equilibria in large non-convex generalized games. Indeed, we construct abstract generalized games where individuals' allocations are bounded. Refereeing to Balder (1999) results of Nash equilibrium existence in generalized games, and to the recent short-proof of its given by Riascos and Torres-Martínez (2010), we assure the existence of equilibrium in our abstract games. After this, using multidimensional Fatou's Lemma, we will prove that Nash equilibria of abstract generalized games converges asymptotically to equilibria of our economy.

The remaining of the paper is organized as follows: In Section 2 we describe the model. The statement of our main result about equilibrium existence is given in Section 3, where we also discuss the assumptions of our model. In Section 4 we discuss different types of garnishment rules compatibles with the framework. Extension of our results are discussed in Section 5. Finally, in the Appendix we make the proof of equilibrium existence.

2. Model

We consider a two period model, without uncertainty at the first period ($t = 0$) and where one state of nature of a finite set S can be reached at the second period ($t = 1$). For convenience of notations, let $S^* = \{0\} \cup S$ be the set of states of nature in the economy, where $s = 0$ denotes the only state of nature at $t = 0$.

There is a finite set L of perfect divisible commodities, which are available for consumption and trade in spot markets at each state of nature. Commodities may be durable between periods $t = 0$ and $t = 1$. That is, there are functions $Y_s : \mathbb{R}^L_+ \to \mathbb{R}^L_+$, with $s \in S$, which represent an exogenous technology that transform bundles that are consumed at the first period in state contingent bundles at the second period. We suppose that, for any $s \in S$, Y_s is the restriction of a linear mapping from $\mathbb{R}^L$ to $\mathbb{R}^L$. We denote by $Y_s(x, \ell)$ the ℓ-th coordinate of vector $Y_s(x) \in \mathbb{R}^L_+$ and by $e(\ell)$ the ℓ-th canonical vector of $\mathbb{R}^L$. Then, if a bundle $x_0 \in \mathbb{R}^L_+$ is consumed at $t = 0$, it transforms into the bundle $Y_s(x_0) \in \mathbb{R}^L_+$ at state $s \in S$. If $Y_s(e(\ell)) = 0$, for any $s \in S$, then we refer to commodity $\ell \in L$ as perishable. When there is at least one state of nature $s \in S$ for which $Y_s(e(\ell), \ell) > 0$, then the commodity ℓ is called durable. However, commodities may transform into other goods and, therefore, they could be neither durable or perishable. That is, a commodity $\ell \in L$ such that both $Y_s(e(\ell)) > 0$ and $Y_s(e(\ell), \ell) = 0$. Let $p_s \in \mathbb{R}^L_+$ be the unitary spot price at state of nature $s \in S^*$ and denote by $p_{s,\ell}$ the unitary price of a commodity ℓ at s. The vector of commodity prices in the economy is denoted by $p = (p_s; s \in S^*)$.

There is a measure space of consumers, $\mathcal{H} = ([0, 1], \mathbb{B}, \mu)$, where $\mathbb{B}$ is the Borel σ-algebra of $[0, 1]$ and μ the Lebesgue measure. Thus, in our economy, each consumer is non-atomic. Agents act on a desire to maximize their utility function using physical and financial markets. Let $w^h_s = (w^h_{s,\ell}; \ell \in L) \in \mathbb{R}^L_+$ be the endowment of commodities that an agent $h \in [0, 1]$ receives at state of nature $s \in S^*$. We denote by $w^h := (w^h_s; s \in S^*)$ the physical endowment plan of agent h. Preferences of an agent h are represented by a utility function $u^h : \mathbb{R}^{L\times S^*}_+ \to \mathbb{R}_+$.

As in Geanakoplos and Zame (2002) or Steinert and Torres-Martínez (2007), there is a finite set J of collateralized debt contracts that can be issued at the first period. When a borrower issues one unit of a debt contract $j \in J$, he receives a quantity of resources π_j and constitutes a physical collateral $C_j \in \mathbb{R}^L_+ \setminus \{0\}$. We denote by $\pi = (\pi_j; j \in J)$ the vector of unitary prices of debt contracts. The vector of state-contingent real promises associated to one unit of debt contract $j \in J$ is given by $(A_{s,j}; s \in S) \in \mathbb{R}^{L\times S}_+$. If the borrower does not honor his promises at a state of nature $s \in S$, the market will seize the associated collateral guarantee and may also implement additional payment enforcement mechanisms.

Each debt contract $j \in J$ is securitized into only one passtrough security. We assume that the unitary price of the security j (the one associated to the debt contract j) is also π_j.[3] Thus, we treat the set of debt contracts and the collection of passtrough securities with the same notation. Let $\theta^h = (\theta^h_j; j \in J) \in \mathbb{R}^J_+$ be the vector of positions of agent h in passtrough securities at $t = 0$. Analogously, $\varphi^h = (\varphi^h_j; j \in J) \in \mathbb{R}^J_+$ denotes the agent h's vector of positions in debt contracts.

Let $x^h = (x^h_s; s \in S^*)$ be the non-collateralized consumption plan for an agent $h \in [0,1]$, where $x^h_s \in \mathbb{R}^L_+$ is the bundle of commodities at state of nature $s \in S^*$ that agent h demand in addition to any collateral guarantee. Particularly, the total consumption plan of agent h at the first period is given by $x^h_0 + \sum_{j\in J} C_j \varphi^h_j \in \mathbb{R}^L_+$.

As we say above, in case of default, agents are burden to deliver the associated collateral bundles. For this reason, an agent $h \in [0,1]$ that borrows φ^h_j units of debt contract $j \in J$ at the first period, delivers at any state of nature $s \in S$ at least an amount of resources $D_{s,j}(p_s)\varphi^h_j$, where $D_{s,j}(p_s) = \min\{p_s A_{s,j}, p_s Y_s(C_j)\}\varphi^h_j$. Thus, the remaining debt of agent h after the strategic decision to pay or foreclosure debts, is given by

$$\Psi_s(p_s, \varphi^h) = \sum_{j\in J} [p_s A_{s,j} - p_s Y_s(C_j)]^+ \varphi^h_j,$$

where $[y]^+ := \max\{y, 0\}$. Additional payment enforcement mechanisms may act over this remaining debt in order to increase the resources that investors receive may act.

In this model, we concentrate our attention in a particular additional enforcement mechanism: the garnishment of individuals wealth in case of bankruptcy. However, we assume that the law protect agents from excessive losses of wealth by confiscation. Specifically, at any state of nature $s \in S$, after the payment or the foreclosure of debts, if some promise remains without fully payment, the legal system does not give to lenders the right to confiscate the entire individual's wealth, since protects a $(1 - \lambda_s) \in (0,1)$ percent of borrower endowment. However, other resources, as the value of either depreciated consumption bundles or financial securities, could be fully garnished.

Thus, given agent' $h \in [0,1]$ consumption and financial decisions at the first period, $(x^h_0, \theta^h, \varphi^h)$, for any state of nature $s \in S$, the maximal amount of resources that agent h may loose if he gives default in at least one of his debts is given by $\Phi^h_s(p_s, R_s, x^h_0, \theta^h, \varphi^h)$, where $R_s = (R_{s,j}; j \in J)$ are the unitary security payments at state of nature s, and $\Phi^h_s : \mathbb{R}^L_+ \times \mathbb{R}^J_+ \times \mathbb{R}^L_+ \times \mathbb{R}^J_+ \times \mathbb{R}^J_+ \to \mathbb{R}_+$ is a

[3] After normalization, it is always possible to make this identification of prices.

continuous function satisfying

$$\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) \leq \lambda_s p_s w_s^h + \left(p_s Y_s(x_0^h) + \sum_{j\in J} [p_s Y_s(C_j) - p_s A_{s,j}]^+ \varphi_j^h + \sum_{j \in J} R_{s,j}\theta_j^h \right).^4$$

It follows that, associated with a debt $\sum_{j\in J} p_s A_{s,j}\varphi_j^h$ at state of nature $s \in S$, an agent $h \in [0,1]$ will strategically decides to pay the following amount of resources,

$$M_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) = \sum_{j\in J} D_{s,j}(p_s)\varphi_j^h + \min\left\{\Psi_s(p_s,\varphi^h), \Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h)\right\}.$$

As we advance above, at each state of nature $s \in S$, an agent $h \in [0,1]$ that invest in θ_j^h units of passtrough security $j \in J$ receives an amount of resources $R_{s,j}\theta_j^h$, where the unitary payments $R_s = (R_{s,j}; j \in J)$ will be determined in equilibrium, since resources payed by debtors over collateral values will be endogenously distributed *pro-rata* to associated investors (i.e. proportional to the seize of original claims).

We assume that, in equilibrium, (i) the quantity of resources that are invested in a passtrough security will coincide with the quantity of resources that are borrowed to the associated debtors, and (ii) the unitary price of a debt-contract coincides with the unitary price of the associated passtrough security. Thus, when a debt contract is traded, the unitary payment of passtrough security j satisfies $D_{s,j}(p_s) \leq R_{s,j}$. That is, in case of default, investors will receive payments that are at least greater than the depreciated collateral guarantees.

Since agents are price takers and also advance unitary security payments, given $(p,\pi,R) \in \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}$, each $h \in [0,1]$ maximize his utility functions by choosing a plan in his budget set $B^h(p,\pi,R)$, which is defined as the set of vectors $(x,\theta,\varphi) \in \mathbb{E} := \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^J$ that satisfies,

$$p_0(x_0 - w_0^h) + \sum_{j\in J} \pi_j(\theta_j - \varphi_j) + p_0 \sum_{j\in J} C_j\varphi_j \leq 0;$$

$$p_s(x_s - w_s^h - Y_s(x_0)) \leq p_s Y_s\left(\sum_{j\in J} C_j\varphi_j\right) + \sum_{j\in J} R_{s,j}\theta_j - M_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h).$$

We denote by $\mathcal{E} = \mathcal{E}\left(S^*, L, (Y_s)_{s\in S}, J, (A_{s,j}, C_j)_{(s,j)\in S\times J}, \mathcal{H}, (u^h, w^h)_{h\in[0,1]}\right)$ our economy with limited-recourse collateralized loans.

Definition 1. *A vector of prices and unitary security payments* $(\overline{p}, \overline{\pi}, \overline{R}) \in \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}$ *jointly with allocations* $\left((\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h); h \in [0,1]\right) \in \mathbb{E}^{[0,1]}$ *constitute an equilibrium of* $\mathcal{E}$ *if the following conditions hold,*

[4] In Section 4 we discuss some examples of garnishment rules that can be captured by our specification of functions $(\Phi_s^h; s \in S)$ and are compatible with the assumptions imposed in our main result below.

(1) *For each* $h \in [0,1]$,

$$u^h\left(\overline{x}_0^h + \sum_{j\in J} C_j \overline{\varphi}_j^h, (\overline{x}_s^h; s \in S)\right) = \max_{(x,\theta,\varphi)\in B^h(\overline{p},\overline{\pi},\overline{R})} u^h\left(x_0 + \sum_{j\in J} C_j \varphi_j, (x_s; s\in S)\right).$$

(2) *Physical and financial markets clear. That is,*

$$\begin{aligned}
\int_{[0,1]} \left(\overline{x}_0^h + \sum_{j\in J} C_j \overline{\varphi}_j^h\right) dh &= \int_{[0,1]} w_0^h dh;\\
\int_{[0,1]} (\overline{x}_s^h - w_s^h) dh &= \int_{[0,1]} Y_s\left(\overline{x}_0^h + \sum_{j\in J} C_j \overline{\varphi}_j^h\right) dh, \quad \forall s \in S;\\
\int_{[0,1]} (\overline{\theta}_j^h - \overline{\varphi}_j^h) dh &= 0, \quad \forall j \in J.
\end{aligned}$$

(3) *At any state of nature* $s \in S$, *aggregate yields equal aggregate payments for any asset* $j \in J$,

$$\overline{R}_{s,j}\int_{[0,1]} \overline{\theta}_j^h dh = \int_{[0,1]} D_{s,j}(\overline{p}_s)\overline{\varphi}_j^h dh + \int_{[0,1]} \beta_s^h(\overline{p}_s, \overline{R}_s, \overline{x}_0^h, \overline{\theta}^h, \overline{\varphi}^h)[\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)]^+ \overline{\varphi}_j^h dh,$$

where $\overline{R}_{s,j} \geq D_{s,j}(\overline{p}_s)$ *and, for any agent* $h \in [0,1]$, *the function*

$$\beta_s^h : \mathbb{R}_+^L \times \mathbb{R}_+^J \times \mathbb{R}_+^L \times \mathbb{R}_+^J \times \mathbb{R}_+^J \to [0,1]$$

is given by

$$\beta_s^h(p_s, R_s, x_0, \theta, \varphi) = \begin{cases} \frac{\Phi_s^h(p_s,R_s,x_0,\theta,\varphi)}{\Psi_s(p_s,\varphi)}, & \textit{when } \Phi_s^h(p_s, R_s, x_0, \theta, \varphi) < \Psi_s(p_s, \varphi);\\ 1, & \textit{in other case.}\end{cases}$$

Note that, by the definition above, if for some $j \in J$, $D_{s,j}(\overline{p}_s) > 0$, then in equilibrium the unitary payments of security j are non-trivial as $\overline{R}_{s,j} > 0$. Analogously to Steinert and Torres-Martínez (2007), we want to assure this property since, in other case, a proof of equilibrium existence may be trivially done. Indeed, if we suppose that both security payments and prices of debt-contract are equal to zero, i.e. $(\overline{\pi}, \overline{R}) = 0$, then agents will not be interested in negotiate financial assets. Thus, any pure spot market equilibrium of the economy without assets is an equilibrium of our economy. For these reasons, using the monotonicity of individuals' preferences and the non-triviality of debt-contract promises (Assumptions (A1) and (A6) below), we assure that the minimum between the original promise and the depreciated collateral value will be effectively strictly positive in equilibrium.

3. Equilibrium existence

Theorem. *Suppose that the following assumptions hold,*

(A1) *For each agent* $h \in [0,1]$, *the utility function* $u^h : \mathbb{R}_+^{L\times S^*} \to \mathbb{R}$ *is continuous and strictly increasing.*

(A2) *Let* $\mathcal{U}(\mathbb{R}_+^{L\times S^*})$ *be the set of functions* $u:\mathbb{R}_+^{L\times S^*}\rightarrow\mathbb{R}^L$ *endowed with the sup norm topology. Then, the mapping* $u:[0,1]\rightarrow\mathcal{U}(\mathbb{R}_+^{L\times S^*})$, *that associates to each agent* $h\in[0,1]$ *the utility function* u^h, *is measurable.*

(A3) *The utility function of any agent* $h\in[0,1]$ *satisfies the following asymptotic property,*

$$\lim_{\sigma\to+\infty} u^h(z_0+\sigma\, C_j,(z_s;s\in S))=+\infty,\quad \forall j\in J,\,\forall(z_s;s\in S^*)\in\mathbb{R}_{++}^{L\times S^*}.$$

(A4) *The function* $w:[0,1]\rightarrow\mathbb{R}_{++}^{L\times S^*}$, *that associated to each* $h\in[0,1]$ *the initial endowment* w^h *is measurable. There exists* $\overline{w}\in\mathbb{R}_+^L$ *such that* $w_s^h\leq\overline{w}$, $\forall(h,s)\in[0,1]\times S^*$.

(A5) *For each* $(h,s)\in[0,1]\times S$, $\lambda_s\in[0,1)$ *and* Φ_s^h *is continuous. Also, given* $(p_s,R_s)\in\mathbb{R}_+^L\times\mathbb{R}_+^J$, *for any agent* $h\in[0,1]$, *the function* $\Phi_s^h(p_s,R_s,\cdot)$ *is convex and has strictly positive values when* $p_s\gg 0$.

(A6) *For each* $j\in J$, *there is a state of nature* $s\in S$ *such that* $\min\{\|A_{s,j}\|_\Sigma\,,\|Y_s(C_j)\|_\Sigma\}>0$.[5]

Then, there exists an equilibrium for our economy.

The first assumption is classical, while the second one is imposed by Riascos and Torres-Martínez (2010) to assure the existence of equilibrium in large non-convex generalized games. Since our technique of proof of equilibrium existence use generalized games too, we need this assumption.

However, Nash equilibria of the generalized games (in the Appendix) not necessarily are equilibria of our economy, because consumption bundles and financial portfolios are truncated in these games (a requirement that our economy does not impose).

To found an equilibrium we will do an asymptotic argument using Fatou's lemma (see Hildenbrand (1974, page 69)). To apply this last result, we need to prove that equilibrium allocations of generalized games are uniformly bounded. We obtain this property as a consequence of the strictly positivity of asymptotic prices and the existence of a uniformly upper bound on individual endowments (Assumption (A4)). Indeed, on the one hand, we will prove that consumption prices are positive due to the strictly monotonicity of utility functions, meanwhile asset prices are strictly positive because asymptotic security payments are non-trivial, which is a consequence of Assumption (A6), as we will show after Lemma 6. On the other hand, the price of the joint operation of taking a loan and constituting the associate collateral bundle is strictly positive. In fact, on one side, since $\lambda_s<1$ (Assumption (A5)), agents will always have an exemption on the amount of resources that may be garnished in case of default at state of nature $s\in S$. On the other hand Assumption (A3) assures that, if an agent may increase his debt without an upper bound, the associated utility level will be unbounded. Thus, when $\overline{p}_0C_j-\overline{\pi}_j\leq 0$, credit is cheaper today and the exemption assures a minimum amount of resources to consume tomorrow. As a consequence of Assumption (A3) there is

[5]The symbol $\|\cdot\|_\Sigma$ denotes the norm of the sum.

no optimal solution for agent's problem. This property allows us to prove that for any asset $j \in J$, $\overline{p}_0 C_j - \overline{\pi}_j > 0$ (see Lemma 7 below). Note that, this happens even when an additional payment enforcement mechanism is introduce: the garnishment of private goods and assets.

Assumptions (A4) and (A5) are also sufficient to prove the lower hemicontinuity of budget set correspondences, which is necessary to assure the existence of equilibrium in generalized games.

4. Examples of garnishment rules

As we said in the model, for any $h \in [0,1]$ there are continuous functions $(\Phi_s^h; s \in S)$ which determine the maximum amount of resources that the law allows to garnish from agent h. However, we know that, independent of the functional form of Φ_s^h, it needs to be strictly less than the total amount of resources that agent h has available at $s \in S$, after the payment and foreclosure of his debts. That is, for any state of nature $s \in S$, there is a $\lambda_s \in [0,1)$ such that,

$$\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) \leq \lambda_s p_s w_s^h + \left(p_s Y_s(x_0^h) + \sum_{j \in J} [p_s Y_s(C_j) - p_s A_{s,j}]^+ \varphi_j^h + \sum_{j \in J} R_{s,j} \theta_j^h \right).$$

Therefore, in addition to the rule that makes Φ_s^h equal to the maximum amount of resources that the law allows to garnish (making the inequality above an equality), we may have the following garnishment rules.

- *Only non-collateralized commodities may be garnished.*

That is, there is a vector $(\zeta_{s,\ell}; \ell \in L) \in (0, \lambda_s]^L$ such that,

$$\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) = \sum_{\ell \in L} \zeta_{s,\ell}\, p_{s,\ell} \left(w_{s,\ell}^h + Y_s(x_0^h, \ell) \right), \quad \forall s \in S.$$

Note that, in case of bankruptcy, the law may protect some commodities more than others.

- *(Almost) only financial investment may be garnished.*

Assume that λ_s is low enough and let

$$\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) = \lambda_s p_s w_s^h + \sum_{j \in J} R_{s,j} \theta_j^h.$$

Note that, since we need at any $s \in S$ a strictly positive $\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h)$ for $p_s \gg 0$ (Assumption (A5)), we can not suppose that only assets are garnished. For this reason we maintain a lower proportion of physical resources as expropriated wealth.

Since λ_s is near to zero, the wealth that can be garnished is closer to the resources obtained as financial investment returns. In some sense, this type of garnishment rule made the additional payment enforcement of our model to be active only over investors, that is, it acts over the richest and most patient agents.

• *The total wealth of defaulters can be garnished.*

This garnishment rule, which is equivalent to take $\lambda_s = 1$ for any $s \in S$, could be incorporated in our model if we strength some of the assumptions of our theorem.

Indeed, we can allow for total garnishment if we suppose that the utility function of each agent $h \in [0,1]$ is separable in time-periods. That is, $u^h((x_s; s \in S^*)) = u^h(0, x_0) + u^h(1, (x_s; s \in S))$. This separability assumption is important to assure that, when $\lambda_s = 1$ for any s, we still have that $\overline{p}_0 C_j - \overline{\pi}_j > 0$ holds for each asset $j \in J$ (see the proof of Lemma 7 in Appendix).

• *A fixed proportional rate of the resources may be garnished.*

It is sufficient to assume that, for any $s \in S$, there is a fixed parameter $\gamma_s \in (0, \lambda_s]$ such that

$$\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) = \gamma_s \left(p_s(w_s^h + Y_s(x_0^h)) + \sum_{j \in J} [p_s Y_s(C_j) - p_s A_{s,j}]^+ \varphi_j^h + \sum_{j \in J} R_{s,j} \theta_j^h \right).$$

• *Survival exemptions in case of bankruptcy.*

Suppose that $\eta \in \mathbb{R}_{++}^L$ is a consumption bundle that measures a threshold that determines, given prices p_s, personalized exemptions. That is, a level of wealth under which is not allowed to garnish more than a minimal percentage $\varepsilon \in (0,1)$ of individual endowments,

$$\Phi_s^h(p_s, R_s, x_0^h, \theta^h, \varphi^h) = \max \left\{ p_s(w_s^h + Y_s(x_0^h)) + \sum_{j \in J} [p_s Y_s(C_j) - p_s A_{s,j}]^+ \varphi_j^h + \sum_{j \in J} R_{s,j} \theta_j^h - p_s \eta; \varepsilon p_s w_s^h \right\}.$$

Then, although in this case the parameter λ_s (that was defined in the model) depends on the wealth of agents, we can maintain the proof of equilibrium, because this parameter still belongs to $[0,1)$, although it increases to one when the wealth of the agent increases.

5. Concluding remarks

We introduced the possibility of bankruptcy into the general equilibrium model with collateralized credit markets of Dubey, Geanakoplos and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007). In case of default, borrowers may loss more than collateral guarantees, as market regulations allow lenders to be reimbursed by the garnishment of debtor wealth. We show that equilibrium always exists when there is a continuum of agents in the economy, even when the garnishment of resources over collateral repossession could induce non-convexities on individuals problems.

Our model can be extended in several dimensions: to allow for more than two periods (or infinite horizon), to introduce other reimbursement rules or additional payment enforcement over the garnishment of wealth, to include financial collateral or even more complex securitization structures. However, we want to highlight two natural questions that may be studied departing for our model.

First, it could be interesting to determine the real effectiveness that the garnishment of wealth has in the process of obtaining higher payments from borrowers. Also, we could analyze its performance relative to another payment enforcement mechanisms, as those given by restrictions on future credit or non-economic punishments that affects utility levels. Secondly, although in our model the garnished wealth is reimbursed to lenders following a proportional rule, we could extend our result to allow some claims to have priority over others to be reimbursed.

In relation with the effectiveness of payment enforcement mechanisms, Ferreira and Torres-Martínez (2010) showed that, in infinite horizon convex economies, the effectiveness of these mechanisms may be incompatible with individual optimality when physical guarantees are low. Indeed, the market value of collateral may be lower than the loan value and, therefore, Ponzi schemes may appear. In our model, which is non-convex, a similar situation may happen. That is, the effectiveness of the garnishment of wealth as payment enforcement may be compromised when the seize of collateral guarantees is low. However, the formalization of these kind of results need to overcome the limitations that non-convexities of our model may generate. On the other hand, to allow for more complicated securitization structures, the same techniques used by Steinert and Torres-Martínez (2007) could be followed.[6]

Appendix: Proof of equilibrium existence

To prove the existence of equilibrium, we will define large non-convex generalized games where (i) each consumer maximizes his utility function, but is restricted to choose *bounded* plans in his budget set; and (ii) there are fictitious players that choose prices and securities payments.

We prove first that those generalized games have equilibria. Secondly, making the upper bound on admissible plans goes to infinity, we find an equilibrium of $\mathcal{E}$ as a cluster point of the sequence of equilibria in generalized games.

Fix $n \in \mathbb{N}$ and define,

$$\begin{aligned}
\mathbb{E}_n &= \{(x,\theta,\varphi) \in \mathbb{E} : (x_{s,\ell},\theta_j,\varphi_j) \leq n(1,1,1),\ \forall (s,\ell,j) \in S^* \times L \times J\},\\
\Delta_0 &= \left\{(q_r; r \in L \cup J) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+ : \sum_{r \in L \cup J} q_r = 1\right\},\\
\Delta_1 &= \left\{(q_r; r \in L) \in \mathbb{R}^L_+ : \sum_{r \in L} q_r = 1\right\}.
\end{aligned}$$

Take as given a vector of prices $(p,\pi) = ((p_0,\pi);(p_s; s \in S)) \in \Delta_0 \times \Delta_1^S$. For convenience of notations, we rewrite unitary payments of a security $j \in J$ at a state of nature $s \in S$ as $R_{s,j} = D_{s,j}(p_s) + N_{s,j}$, where $N_{s,j} \geq 0$. Thus, let $N = (N_{s,j}; (s,j) \in S \times J) \in [0,\overline{A}]^{S \times J}$ be the vector of contingent security payments over collateral values, where $\overline{A} := \max_{(s,j) \in S \times J} \sum_{\ell \in L} A_{s,j,\ell}$. The

[6] Essentially, following Steinert and Torres-Martínez (2007) we can change the specification of the large generalized game in the Appendix, in order to include seniority structures of reimbursement, maintaining the equilibrium existence result.

truncated budget set of agent $h \in [0,1]$, denoted by $B_n^h(p,\pi,N)$, is defined as the collection of of plans $(x_n,\theta_n,\varphi_n,\alpha_n,\kappa_n) \in \mathbb{E}_n \times [0,1]^S \times [0,n]^S$ that satisfies,

$$p_0 x_{n,0} + \sum_{j\in J} \pi_j(\theta_{n,j} - \varphi_{n,j}) + p_0 \sum_{j\in J} C_j \varphi_{n,j} \leq p_0 w_0^h;$$

$$\begin{aligned} p_s(x_{n,s} - w_s^h) \leq p_s Y_s \left(x_{n,0} + \sum_{j\in J} C_j \varphi_{n,j} \right) &+ \sum_{j\in J} D_{s,j}(p_s)\,(\theta_{n,j} - \varphi_{n,j}) \\ &+ \sum_{j\in J} N_{s,j}\theta_{n,j} - (\alpha_{n,s}\Psi_s(p_s,\varphi_n) + \kappa_{n,s}), \quad \forall s \in S; \end{aligned}$$

$$\alpha_{n,s}\Psi_s(p_s,\varphi_n) + \kappa_{n,s} \geq \min\{\Psi_s(p_s,\varphi_n), \Phi_s^h(p_s, N_s + D_s(p_s), x_{n,0}, \theta_n, \varphi_n)\}, \quad \forall s \in S.$$

We introduce the auxiliary variables $((\alpha_{n,s},\kappa_{n,s}); s \in S)$ in order to prove equilibrium existence in our generalized large games (that we will define below). Essentially, we will need that the objective functions of fictitious players depends on aggregated information about the actions of consumers, but also that this aggregated information does not depends on prices (as would be the case if we work with variables $M_s^h(p_s, N_s + D_s(p_s), x_{n,0}, \theta_n, \varphi_n)$. Additionally, although the introduction of variables $(\alpha_{n,s}; s \in S)$ is sufficient to attempt this objective, variables $(\kappa_{n,s}; s \in S)$ allow us to prove that truncated budget set correspondences are lower-hemicontinuous (see Lemma 1 below).

The generalized game $\mathcal{G}_n$. Given $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a generalized game with a continuum of players, where only a finite number of them are atomic. In this game, the set of players jointly with their actions spaces, admissible strategies and objective functions, may be described as follows,

(a) Given a vector of prices and payments $(p,\pi,N) \in \Delta_0 \times \Delta_1^S \times [0,\overline{A}]^{S\times J}$, each consumer $h \in [0,1]$ maximizes the function $v_n^h : \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times [0,n]^S \to \mathbb{R}_+$,

$$v_n^h(x_n^h, \varphi_n^h, \kappa_n^h) = u^h \left(x_{n,0}^h + \sum_{j\in J} C_j \varphi_j^h, (x_{n,s}^h; s \in S) \right) - \sum_{s\in S} \kappa_{n,s}^h,$$

by choosing a plan $(x_n^h, \theta_n^h, \varphi_n^h, \alpha_n^h, \kappa_n^h) \in B_n^h(p,\pi,N)$.

Define the continuous function $\tau : \mathbb{E}_n \times [0,1]^S \times [0,n]^S \to \mathbb{E}_n \times [0,n]^{S\times J}$ by $\tau(x,\theta,\varphi,\alpha,\kappa) = (x,\theta,\varphi,\alpha\odot\varphi)$, where for each vector $(\alpha,\varphi) \in [0,1]^S \times [0,n]^J$, $\alpha\odot\varphi = (\alpha_s\varphi_j; (s,j) \in S\times J) \in \mathbb{R}_+^{S\times J}$. Let $\mathcal{F}_n$ be the set of action profiles for players $h \in [0,1]$, that is, the set of functions $f : [0,1] \to \mathbb{E}_n \times [0,1]^S \times [0,n]^S$.

In addition to consumers $h \in [0,1]$, we include in the generalized game $\mathcal{G}_n$ players that take as given messages about the actions taken by the consumers. The set of messages is given by,

$$\text{Mess}_n = \left\{ \int_{[0,1]} \tau(f(h))dh : \; (f \in \mathcal{F}_n) \;\; \wedge \;\; (\tau \circ f \text{ is measurable}) \right\}.$$

Then, in addition to players $h \in [0,1]$, we have,

(b) A player a_0 that, given $m \in \text{Mess}_n$, chooses a vector of prices $(p_0, \pi) \in \Delta_0$ in order to maximize the function

$$p_0 \int_{[0,1]} \left(x^h_{n,0} + \sum_{j \in J} C_j \varphi^h_{n,j} - w^h_0 \right) dh + \sum_{j \in J} \pi_j \int_{[0,1]} \left(\theta^h_{n,j} - \varphi^h_{n,j} \right) dh,$$

where $m = \int_{[0,1]} (x^h_n, \theta^h_n, \varphi^h_n, \alpha^h_n \odot \varphi^h_n) dh$.

(c) For any $s \in S$, a player a_s that, given $m \in \text{Mess}_n$, chooses a vector of prices $p_s \in \Delta_1$ in order to maximize the function

$$p_s \int_{[0,1]} \left(x^h_{n,s} - w^h_s - Y_s \left(x^h_{n,0} + \sum_{j \in J} C_j \varphi^h_{n,j} \right) \right) dh,$$

where $m = \int_{[0,1]} (x^h_n, \theta^h_n, \varphi^h_n, \alpha^h_n \odot \varphi^h_n) dh$.

(d) For each pair $(s,j) \in S \times J$, a player $c_{s,j}$ that, given $(m, p_s) \in \text{Mess}_n \times \Delta_1$, chooses $N_{s,j} \in [0, \overline{A}]$ in order to maximize the function

$$-\left(N_{s,j} \int_{[0,1]} \varphi^h_j dh - \int_{[0,1]} [p_s A_{s,j} - p_s Y_s(C_j)]^+ \alpha^h_s \varphi^h_j dh \right)^2,$$

where $m = \int_{[0,1]} (x^h_n, \theta^h_n, \varphi^h_n, \alpha^h_n \odot \varphi^h_n) dh$.

Definition 2. *An Nash equilibrium in pure strategies for the game $\mathcal{G}_n$ is given by a plan of strategies and a message*

$$\left(\left((\overline{p}^n_0, \overline{\pi}^n), \overline{p}^n_s, \overline{N}^n_{s,j} \right)_{(s,j) \in S \times J}; \left(\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n, \overline{\alpha}^h_n, \overline{\kappa}^h_n \right)_{h \in [0,1]}, \overline{m} \right),$$

such that, any player maximizes his objective function given the message and the strategies chosen by the other players, where $\overline{m} = \int_{[0,1]} (\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n, \overline{\alpha}^h_n \odot \overline{\varphi}^h_n) dh$.

Lemma 1. *Under Assumptions (A1), (A2), (A4) and (A5), there exists $n^* \in \mathbb{N}$ such that, for any $n > n^*$, there is a pure strategy Nash equilibrium of the generalized game $\mathcal{G}_n$.*

Proof. The existence of a pure strategy equilibrium in our game is a consequence of Theorem 1 in Riascos and Torres-Martínez (2010) (see also Balder (1999)). The only requirement of this Theorem that does not follows from simple arguments or direct verification, is the lower-hemicontinuity of correspondences B^h_n, for any $h \in [0,1]$.

Thus, given $h \in [0,1]$, consider the correspondence $\dot{B}_n^h$ that associates to a vector (p,π,N) the collection of plans $(x_n,\theta_n,\varphi_n,\alpha_n,\kappa_n) \in \mathbb{E}_n \times [0,1]^S \times [0,n]^S$ that satisfies,

$$p_0 x_{n,0} + \sum_{j\in J} \pi_j(\theta_{n,j} - \varphi_{n,j}) + p_0 \sum_{j\in J} C_j \varphi_{n,j} < p_0 w_0^h;$$

$$\begin{aligned} p_s(x_{n,s} - w_s^h) < p_s Y_s \left(x_{n,0} + \sum_{j\in J} C_j \varphi_{n,j} \right) + \sum_{j\in J} D_{s,j}(p_s)\,(\theta_{n,j} - \varphi_{n,j}) \\ + \sum_{j\in J} N_{s,j}\theta_{n,j} - (\alpha_{n,s}\Psi_s(p_s,\varphi_n) + \kappa_{n,s}), \quad \forall s \in S; \end{aligned}$$

$$\alpha_{n,s}\Psi_s(p_s,\varphi_n) + \kappa_{n,s} > \min\{\Psi_s(p_s,\varphi_n), \Phi_s^h(p_s, N_s + D_s(p_s), x_{n,0}, \theta_n, \varphi_n)\}, \quad \forall s \in S.$$

It follows from Assumption (A4) that $\dot{B}_n^h$ has non-empty values. Also, since the constraints that define $\dot{B}_n^h(p,\pi,N)$ are given by inequalities that only include continuous functions, the correspondence $\dot{B}_n^h$ has open graph. Therefore, for any $h \in [0,1]$, $\dot{B}_n^h$ is lower-hemicontinuous (see Hildenbrand (1974, Theorem 2, page 27)). Moreover, the correspondence that associates to any vector (p,π,N) the closure of the set $\dot{B}_n^h(p,\pi,N)$ is also lower-hemicontinuous (see Hildenbrand (1974, page 26)).

We affirm that, for any vector $(p,\pi,N) \in \Delta_0 \times \Delta_1^S \times [0,\overline{A}]^{S\times J}$, the closure of the set $\dot{B}_n^h(p,\pi,N)$ coincides with $B_n^h(p,\pi,N)$. Since, by construction, closure$(\dot{B}_n^h(p,\pi,N)) \subset B_n^h(p,\pi,N)$, it is sufficient to prove that, $B_n^h(p,\pi,N) \subset$ closure$(\dot{B}_n^h(p,\pi,N))$.

Therefore, fix $(x_n,\theta_n,\varphi_n,\alpha_n,\kappa_n) \in B_n^h(p,\pi,N) \subset \mathbb{E}_n \times [0,1]^S \times [0,n]^S$.

Given $((\epsilon_s; s \in S^*),\delta) \in (0,1)^{S^*} \times (0,1)$, for any $j \in J$, define $\varphi_{n,j}(\epsilon_0,\delta) = (1-\delta)\varphi_{n,j} + \epsilon_0$ and, for any $s \in S$, let $\kappa_{n,s}(\epsilon_s,\delta) = (1-\delta)\kappa_{n,s} + \epsilon_s$. We want to prove that the plan

$$((1-\delta)x_n, (1-\delta)\theta_n, (\varphi_{n,j}(\epsilon_0,\delta))_{j\in J}, \alpha_n, (\kappa_{n,s}(\epsilon_s,\delta))_{s\in S})$$

belongs to the interior of $B_n^h(p,\pi,N)$ (i.e. constraints are satisfied with strictly inequality).

However, it is not difficult to verify that this property effectively holds if $n > n^* := \max_{\ell\in L} \overline{w}_\ell$, and the following inequalities are satisfied by the parameters $((\epsilon_s; s \in S^*),\delta)$, [7]

$$\begin{aligned} &\Phi_s^h(p_s, N_s + D_s(p_s), (1-\delta)x_{n,0}, (1-\delta)\theta_n, \varphi_n(\epsilon_0,\delta)) \\ &\qquad\qquad < (1-\delta)\Phi_s^h(p_s, N_s + D_s(p_s), x_{n,0}, \theta_n, \varphi_n) + \epsilon_s, \end{aligned}$$

$$\epsilon_s < \min\left\{ \delta p_s w_s^h - \sum_{j\in J} p_s A_{s,j}\epsilon_0, \delta n \right\} = \delta\, p_s w_s^h - \sum_{j\in J} p_s A_{s,j}\epsilon_0, \quad \forall s \in S;$$

$$\epsilon_0 \sum_{j\in J} (p_0 C_j - \pi_j) < \delta \min_{\ell\in L} w_{0,\ell}^h.$$

[7]Remember that, for any $(p_s,R_s) \in \mathbb{R}_+^L \times \mathbb{R}_+^J$, the function $\Phi_s^h(p_s,R_s,x_0,\varphi,\theta)$ is convex on (x_0,θ,φ) and $\Phi_s^h(p_s,R_s,0,0,0) < p_s w_s^h$ for any agent $h \in [0,1]$. On the other hand, the restriction over n is to assure that agents have freedom to consume their entire physical endowment in any state of nature.

Note that, inequalities above are well defined as a consequence of Assumption (A4). Thus, making δ goes to zero (which implies that $(\epsilon_s; s \in S^*)$ vanishes too), we conclude that $(x_n, \theta_n, \varphi_n, \alpha_n, \kappa_n)$ belong to the closure of $\dot{B}^h_n(p, \pi, N)$.

Thus, if $n > n^*$, B^h_n is lower-hemicontinuous for every agent $h \in [0, 1]$. □

In any equilibrium of the game $\mathcal{G}_n$, with $n > n^*$,

$$\left(\left((\overline{p}^n_0, \overline{\pi}^n), \overline{p}^n_s, \overline{N}^n_{s,j}\right)_{(s,j)\in S\times J}; \left(\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n, \overline{\alpha}^h_n, \overline{\kappa}^h_n\right)_{h\in[0,1]}, \overline{m}\right),$$

each consumer $h \in [0, 1]$ will choose $\overline{\kappa}^h_n = 0$. In fact, the variable $\kappa_{n,s}$ reduces the income of the agent at $s \in S$ and also generates a penalty in the utility. Thus, as a consequence of monotonicity of preferences (Assumption (A1)), the agent does not have any incentive to make $\overline{\kappa}^h_{n,s} > 0$, since $\alpha_{n,s} \in [0, 1]$. Thus, it follows that, for any agent $h \in [0, 1]$, $\overline{\alpha}^h_{n,s} = \beta^h_s(\overline{p}^n_s, \overline{N}^n_s + D_s(\overline{p}^n_s), \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)$. Indeed, since preferences are monotonic (Assumption (A1)), agent h will never choose, at a state of nature $s \in S$, an $\overline{\alpha}^h_{n,s}\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n) > \min\left\{\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n), \Phi^h_s(\overline{p}^n_s, \overline{N}^n_s + D_s(\overline{p}^n_s), \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)\right\}$.

Lemma 2. *Suppose that Assumptions (A1), (A2), (A4) and (A5) hold. Then, for any $n > n^*$, given a Nash equilibrium of $\mathcal{G}_n$,*

$$\left(\left((\overline{p}^n_0, \overline{\pi}^n), \overline{p}^n_s, \overline{N}^n_{s,j}\right)_{(s,j)\in S\times J}; \left(\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n, \overline{\alpha}^h_n, \overline{\kappa}^h_n\right)_{h\in[0,1]}, \overline{m}\right),$$

for each $(s, j) \in S \times J$,

$$\overline{N}^n_{s,j}\int_{[0,1]} \overline{\varphi}^h_{n,j} dh = \int_{[0,1]} [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+ \overline{\alpha}^h_{n,s}\overline{\varphi}^h_{n,j} dh.$$

Proof. Let $n > n^*$ and fix $(s, j) \in S \times J$. Since $\overline{N}^n_{s,j} \in [0, \overline{A}]$, it follows from the definition of the objective function of player $c_{s,j}$ that,

$$\overline{N}^n_{s,j}\int_{[0,1]} \overline{\varphi}^h_{n,j} dh \le \int_{[0,1]} [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+ \overline{\alpha}^h_{n,s}\overline{\varphi}^h_{n,j} dh,$$

where the strict inequality holds only if both $\overline{N}^n_{s,j} = \overline{A}$ and $\int_{[0,1]} \overline{\varphi}^h_{n,j} dh > 0$, but this is impossible since $\overline{p}^n_s \in \Delta_1$. Thus, the equality always holds. □

Definition 3. *A vector of prices and payments $(\overline{p}^n, \overline{\pi}^n, \overline{R}^n) \in \Delta_0 \times \Delta_1^S \times [0, 2\overline{A}]^{S\times J}$, jointly with plans $\left((\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n); h \in [0, 1]\right) \in \mathbb{E}^{[0,1]}_n$, constitute a n-equilibrium of $\mathcal{E}$ when,*

(3.1) *For each $h \in [0, 1]$,*

$$(\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n) \in argmax_{(x,\theta,\varphi)\in B^h(\overline{p}^n, \overline{\pi}^n, \overline{R}^n)\cap\mathbb{E}_n}\ u^h\left(x_0 + \sum_{j\in J} C_j\varphi_j, (x_s; s \in S)\right).$$

(3.2) *There is no excess of demand in physical or financial markets,*

$$\begin{aligned}
\int_{[0,1]} \left(\overline{x}_{n,0}^h + \sum_{j\in J} C_j \overline{\varphi}_{n,j}^h \right) dh &\leq \int_{[0,1]} w_0^h dh; \\
\int_{[0,1]} (\overline{x}_{n,s}^h - w_s^h) dh &\leq \int_{[0,1]} Y_s \left(\overline{x}_{n,0}^h + \sum_{j\in J} C_j \overline{\varphi}_{n,j}^h \right) dh, \quad \forall s \in S; \\
\int_{[0,1]} (\overline{\theta}_{n,j}^h - \overline{\varphi}_{n,j}^h) dh &\leq 0, \quad \forall j \in J.
\end{aligned}$$

(3.3) *At any state of nature $s \in S$ and for any $j \in J$,*

$$\begin{aligned}
&\overline{R}_{s,j}^n \int_{[0,1]} \overline{\theta}_{n,j}^h dh \leq \overline{R}_{s,j}^n \int_{[0,1]} \overline{\varphi}_{n,j}^h dh \\
&\qquad = \int_{[0,1]} D_{s,j}(\overline{p}_s^n) \overline{\varphi}_{n,j}^h dh + \int_{[0,1]} [\overline{p}_s^n A_{s,j} - \overline{p}_s^n Y_s(C_j)]^+ \beta_s^h(\overline{p}_s^n, \overline{R}_s^n, \overline{x}_{n,0}^h, \overline{\theta}_n^h, \overline{\varphi}_n^h) \overline{\varphi}_{n,j}^h dh,
\end{aligned}$$

where $\overline{R}_s^n = (\overline{R}_{s,j}^n; j \in J)$ and $\overline{R}_{s,j}^n \geq D_{s,j}(\overline{p}_s^n)$.

The following result assures the existence of n-equilibria as a consequence of the existence of Nash equilibria in the generalized game $\mathcal{G}_n$.

Lemma 3. *Under Assumptions (A1), (A2), (A4) and (A5), the economy $\mathcal{E}$ has a n-equilibrium for any $n > n^*$.*

Proof. Given $n > n^*$, let

$$\left(\left((\overline{p}_0^n, \overline{\pi}^n), \overline{p}_s^n, \overline{N}_{s,j}^n \right)_{(s,j)\in S\times J}; \left(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h, \overline{\alpha}_n^h, \overline{\kappa}_n^h \right)_{h\in[0,1]}, \overline{m} \right),$$

be a pure strategy Nash equilibrium of $\mathcal{G}_n$. We want to prove that

$$\left(\left((\overline{p}_0^n, \overline{\pi}^n), \overline{p}_s^n, \overline{R}_{s,j}^n \right)_{(s,j)\in S\times J}; \left(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h \right)_{h\in[0,1]} \right)$$

constitutes a n-equilibrium for the economy $\mathcal{E}$, where for each $(s,j) \in S \times J$ the unitary security payment satisfies, $\overline{R}_{s,j}^n = D_{s,j}(\overline{p}_s^n) + \overline{N}_{s,j}^n$. It follows from comments after Lemma 1, that to attempt this objective is sufficient to prove that conditions of items (3.2) and (3.3) of Definition 3 hold.

Integrating through agents the first period budget constraints of $B_n^h(\overline{p}^n, \overline{\pi}^n, \overline{N}^n)$, we obtain that,

$$\overline{p}_0^n \int_{[0,1]} \left(\overline{x}_{n,0}^h + \sum_{j\in J} C_j \overline{\varphi}_{n,j}^h - w_0^h \right) dh + \sum_{j\in J} \overline{\pi}_j^n \int_{[0,1]} \left(\overline{\theta}_{n,j}^h - \overline{\varphi}_{n,j}^h \right) dh \leq 0.$$

Thus, the maximal value of player a_0 objective function is zero. Therefore, since $(\overline{p}_0^n, \overline{\pi}^n) \in \Delta_0$, for any commodity $\ell \in L$ and for each $j \in J$,

$$\int_{[0,1]} \left(\overline{x}_{n,0,\ell}^h + \sum_{j\in J} C_{j,\ell} \overline{\varphi}_{n,j}^h - w_{0,\ell}^h \right) dh \leq 0, \qquad \int_{[0,1]} \left(\overline{\theta}_{n,j}^h - \overline{\varphi}_{n,j}^h \right) dh \leq 0.$$

In fact, in other case, player a_0 would make his objective function positive by concentrating in those coordinates that are strictly positive. Thus, as a direct consequence of the last inequality above and Lemma 2, we obtain

$$\overline{N}^n_{s,j}\int_{[0,1]}\overline{\theta}^h_{n,j}dh \le \overline{N}^n_{s,j}\int_{[0,1]}\overline{\varphi}^h_{n,j}dh = \int_{[0,1]}[\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+\overline{\alpha}^h_{n,s}\overline{\varphi}^h_{n,j}dh.$$

If we define $\overline{R}^n_{s,j} = D_{s,j}(\overline{p}^n_s) + \overline{N}^n_{s,j}$, we obtain conditions of item (3.3) of Definition 3, using the fact that $\overline{\alpha}^h_{n,s} = \beta^h_s(\overline{p}^n_s, \overline{R}^n_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)$.

Finally, given $s \in S$, using inequalities above and aggregating budget constraints at this state of nature, we obtain that

$$\overline{p}^n_s\left(\int_{[0,1]}(\overline{x}^h_{n,s} - w^h_s)dh - \int_{[0,1]}Y_s\left(\overline{x}^h_{n,0} + \sum_{j\in J}C_j\overline{\varphi}^h_{n,j}\right)dh\right) \le 0.$$

In other words, the maximal value of the objective function of player a_s is less than or equal to zero. Therefore, since $\overline{p}^n_s$ belongs to Δ_1, we conclude that, for any commodity $\ell \in L$

$$\int_{[0,1]}(\overline{x}^h_{n,s,\ell} - w^h_{s,\ell})dh \le \int_{[0,1]}Y_s\left(\overline{x}^h_{n,0,\ell} + \sum_{j\in J}C_{j,\ell}\overline{\varphi}^h_{n,j}\right)dh.$$

Thus, $\Big((\overline{p}^n_0, \overline{\pi}^n), \overline{p}^n_s, \overline{R}^n_{s,j})_{(s,j)\in S\times J}; (\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n)_{h\in[0,1]}\Big)$ constitutes a n-equilibrium of $\mathcal{E}$. □

Lemma 4. *Under Assumptions (A1), (A2), (A4) and (A5), let* $\Big((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \Big((\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n); h \in [0,1]\Big)\Big)$ *be a n-equilibrium of $\mathcal{E}$, with $n > n^*$. Consider the family of non-negative and integrable functions* $\Big\{g_n : [0,1] \to \mathbb{R}^{L\times S^*}_+ \times \mathbb{R}^J_+ \times \mathbb{R}^J_+ \times \mathbb{R}^{S\times J}_+\Big\}_{n>n^*}$ *given by,*

$$g_n(h) = \Big(\overline{x}^h_n, \overline{\theta}^h_n,\, \overline{\varphi}^h_n,\, (\beta^h_s(\overline{p}^n_s, \overline{R}^n_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)\,\overline{\varphi}^h_{n,j})_{(s,j)\in S\times J}\Big),\ \ \forall n > n^*.$$

Then, the sequence $\Big\{\Big(\overline{p}^n, \overline{\pi}^n, \overline{R}^n, \int_{[0,1]} g_n(h)dh\Big)\Big\}_{n>n^*}$ *is bounded and, therefore, has a convergent subsequence.*

Proof. Since for any $n > n^*$, the vector $(\overline{p}^n, \overline{\pi}^n, \overline{R}^n) \in \Delta_0 \times \Delta^S_1 \times [0, 2\overline{A}]^{S\times J}$, it follows that the sequence of equilibrium prices and payments is bounded. On the other hand, using the fact that $\Big((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \Big((\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n); h \in [0,1]\Big)\Big)$ is an n-equilibrium of $\mathcal{E}$ we have,

$$0 \le \int_{[0,1]}\overline{x}^h_{n,0}dh \le \int_{[0,1]}w^h_0 dh,$$

$$0 \le \sum_{j\in J}C_j\int_{[0,1]}\overline{\varphi}^h_{n,j}dh = \int_{[0,1]}\sum_{j\in J}C_j\overline{\varphi}^h_{n,j}dh \le \int_{[0,1]}w^h_0 dh,$$

$$0 \le \int_{[0,1]}\overline{\theta}^h_n dh \le \int_{[0,1]}\overline{\varphi}^h_n dh.$$

Moreover, for any $(s,j) \in S \times J$,

$$0 \leq \int_{[0,1]} \beta_s^h(\overline{p}_s^n, \overline{R}_s^n, \overline{x}_{n,0}^h, \overline{\theta}_n^h, \overline{\varphi}_n^h)\, \overline{\varphi}_{n,j}^h dh \leq \int_{[0,1]} \overline{\varphi}_{n,j}^h dh,$$

$$0 \leq \int_{[0,1]} \overline{x}_{n,s}^h dh \leq \int_{[0,1]} \left(w_s^h + Y_s\left(w_0^h\right)\right) dh,$$

where the last inequality is a consequence of the fact that $Y(x) \leq Y(y)$ if $x \leq y$. The result follows from Assumption (A4), since for any $j \in J$ there is $\ell \in L$ such that $C_{j,\ell} > 0$. □

It follows from Lemma above that, if we fix a sequence of n-equilibria

$$\left\{\left((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \left((\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h); h \in [0,1]\right)\right)\right\}_{n>n^*},$$

there exists a convergent subsequence

$$\left\{\left(\overline{p}^{n_k}, \overline{\pi}^{n_k}, \overline{R}^{n_k}, \int_{[0,1]} g_{n_k}(h)dh\right)\right\}_{n_k>n^*} \subseteq \left\{\left(\overline{p}^n, \overline{\pi}^n, \overline{R}^n, \int_{[0,1]} g_n(h)dh\right)\right\}_{n>n^*}.$$

We denote by $(\overline{p}, \overline{\pi}, \overline{R})$ the associated limit of prices and payments. Also, applying the weak version of the multidimensional Fatou's Lemma to the sequence $\{g_{n_k}\}_{n_k>n^*}$ (see Hildenbrand (1974, page 69)), we can found a set $\mathbb{P} \subset [0,1]$ of full measure ($\mu(\mathbb{P}) = 1$), and an integrable function $g : [0,1] \to \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}$, defined by $g(h) := (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, (\overline{\rho}_{s,j}^h)_{(s,j)\in S\times J})$ such that, for each agent $h \in \mathbb{P}$, there is a subsequence of $\{g_{n_k}(h)\}_{n_k>n^*}$ that converges to $g(h)$, and

$$\int_{[0,1]} g(h)dh \leq \lim_{k\to\infty} \int_{[0,1]} g_{n_k}(h)dh.$$

Thus, it follows that, for any $h \in \mathbb{P}$, the bundle $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)$ belong to $B^h(\overline{p}, \overline{\pi}, \overline{R})$. In addition, if commodity prices satisfy $\overline{p}_s \gg 0$, then for any $(h,s,j) \in \mathbb{P} \times S \times J$, we have that $\overline{\rho}_{s,j}^h = \beta_s^h(\overline{p}_s, \overline{R}_s, \overline{x}_0^h, \overline{\theta}^h, \overline{\varphi}^h)\, \overline{\varphi}_j^h$.[8]

Lemma 5. *Under Assumptions (A1), (A2), (A4) and (A5), take as given $n > n^*$ and $h \in [0,1]$. Then, the correspondence $\mathcal{H}_n^h : \Delta_0 \times \Delta_1 \times [0, 2\overline{A}]^{S\times J} \to \mathbb{E}_n$ defined by $\mathcal{H}_n^h(p,\pi,R) = B^h(p,\pi,R) \cap \mathbb{E}_n$ is lower-hemicontinuous.*

[8]Given $h \in \mathbb{P}$, the convergence of a subsequence of $\{\overline{\varphi}_{n_k}^h, (\beta_s^h(\overline{p}_s^{n_k}, \overline{R}_s^{n_k}, \overline{x}_{n_k,0}^h, \overline{\theta}_{n_k}^h, \overline{\varphi}_{n_k}^h)\, \overline{\varphi}_{n_k,j}^h)_{(s,j)\in S\times J}\}_{n_k>n^*}$ (those given by the Fatou's Lemma), does not necessarily imply in the convergence of the associated subsequence of $(\beta_s^h(\overline{p}_s^{n_k}, \overline{R}_s^{n_k}, \overline{x}_{n_k,0}^h, \overline{\theta}_{n_k}^h, \overline{\varphi}_{n_k}^h)_{s\in S}\}_{n_k>n^*}$. However, the later sequence is bounded and, therefore, taking a subsequence again if it is necessary, we can assume that its converges. Thus, if $\Psi_s(\overline{p}_s, \overline{\varphi}^h) > 0$, then for any $\overline{p}_s \in \Delta_1$, the function β_s^h is continuous at the point $(\overline{p}_s, \overline{R}_s, \overline{x}_0^h, \overline{\theta}^h, \overline{\varphi}^h)$ and, therefore, $\overline{\rho}_{s,j}^h = \beta_s^h(\overline{p}_s, \overline{R}_s, \overline{x}_0^h, \overline{\theta}^h, \overline{\varphi}^h)\, \overline{\varphi}_j^h$. When $\overline{p}_s \gg 0$, if $\Psi_s(\overline{p}_s, \overline{\varphi}^h) = 0$, then the fact that $\Phi_s^h(\overline{p}_s, \overline{R}_s, \overline{x}_0^h, \overline{\theta}^h, \overline{\varphi}^h) > 0$ jointly with the continuity of Ψ_s assure that, for n_k large enough, $\beta_s^h(\overline{p}_s^{n_k}, \overline{R}_s^{n_k}, \overline{x}_{n_k,0}^h, \overline{\theta}_{n_k}^h, \overline{\varphi}_{n_k}^h) = 1$, which implies that $\overline{\rho}_{s,j}^h = \beta_s^h(\overline{p}_s, \overline{R}_s, \overline{x}_0^h, \overline{\theta}^h, \overline{\varphi}^h)\, \overline{\varphi}_j^h$.

PROOF. We follow similar arguments to those made in Lemma 1 to prove the lower-hemicontinuity of the correspondence B_n^h. Indeed, since the associated interior correspondence $\dot{\mathcal{H}}_n^h$ has non-empty values and open graph, it follows that both correspondences $\dot{\mathcal{H}}_n^h$ and closure($\dot{\mathcal{H}}_n^h$) are lower-hemicontinuous. Thus, since for any vector (p,π,R), $\dot{\mathcal{H}}_n^h(p,\pi,R)\subset\mathcal{H}_n^h(p,\pi,R)$, only left to prove that $\mathcal{H}_n^h(p,\pi,R)\subset$ closure($\dot{\mathcal{H}}_n^h(p,\pi,R)$).

Given a vector of prices and payments (p,π,R), fix $(x_n,\theta_n,\varphi_n)\in\mathcal{H}_n^h(p,\pi,R)$. For any $(\delta,\epsilon_0)\in(0,1)\times(0,1)$, define $\varphi_{n,j}(\epsilon_0,\delta)=(1-\delta)\varphi_{n,j}+\epsilon_0$, $\forall j\in J$. Then, if the following conditions hold

$$\epsilon_0\sum_{j\in J}(p_0C_j-\pi_j)<\delta\min_{\ell\in L}w_{0,\ell}^h,$$

$$\Phi_s^h(p_s,R_s,(1-\delta)x_{n,0},(1-\delta)\theta_n,\varphi_n(\epsilon_0,\delta))$$

$$<(1-\delta)\Phi_s^h(p_s,R_s,x_{n,0},\theta_n,\varphi_n)+\delta\,p_sw_s^h-\sum_{j\in J}p_sA_{s,j}\epsilon_0,$$

the constraints on $B^h(p,\pi,N)\cap\mathbb{E}_n$ are satisfied with strict inequality by the plan $((1-\delta)x_n,(1-\delta)\theta_n,(\varphi_{n,j}(\epsilon_0,\delta))_{j\in J})$. In this way, if δ goes to zero (which implies that ϵ_0 vanishes too), we conclude that (x_n,θ_n,φ_n) belong to the closure of $\dot{\mathcal{H}}_n^h(p,\pi,R)$. Thus, $\mathcal{H}_n^h$ is lower-hemicontinuous. □

LEMMA 6. *Under Assumptions (A1), (A2), (A4) and (A5), for each agent $h\in\mathbb{P}$, the allocation $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$ is an optimal choice on $B^h(\overline{p},\overline{\pi},\overline{R})$.*

PROOF. Fix an agent $h\in\mathbb{P}$ and suppose that $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$ is not optimal for agent h at prices-payments $(\overline{p},\overline{\pi},\overline{R})$. Thus, there exists another plan $(\tilde{x}^h,\tilde{\theta}^h,\tilde{\varphi}^h)\in B^h(\overline{p},\overline{\pi},\overline{R})$ such that

$$u^h\left(\tilde{x}_0^h+\sum_{j\in J}C_j\tilde{\varphi}_j^h,\left(\tilde{x}_s^h;s\in S\right)\right)>u^h\left(\overline{x}_0^h+\sum_{j\in J}C_j\overline{\varphi}_j^h,\left(\overline{x}_s^h;s\in S\right)\right).$$

It is clear that there exists $n^{**}>n^*$ such that, for any $n\geq n^{**}$ plans $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$ and $(\tilde{x}^h,\tilde{\theta}^h,\tilde{\varphi}^h)$ belongs to $\mathcal{H}_n^h(\overline{p},\overline{\pi},\overline{R})\supseteq\mathcal{H}_{n^*}^h(\overline{p},\overline{\pi},\overline{R})$.

Fix $n>n^{**}$. Then, there exists an $T_n^h\in\mathbb{N}$ such that $(\overline{x}_m^h,\overline{\theta}_m^h,\overline{\varphi}_m^h)\in\mathbb{E}_n$, for any $m>T_n^h$ in the subsequence of $\{g_{m'}(h)\}_{m'>n^*}$ that was given by the Fatou's Lemma and converges to $g(h)$.

It follows from Lemma 5 and the sequential characterization of lower-hemicontinuity, that there exist a sequence $\left\{(\tilde{x}_m^h,\tilde{\theta}_m^h,\tilde{\varphi}_m^h)\right\}_{m>T_n^h}\in\mathbb{E}_n$ such that, for any $m>T_n^h$, the plan $(\tilde{x}_m^h,\tilde{\theta}_m^h,\tilde{\varphi}_m^h)\in\mathcal{H}_n^h(\overline{p}^m,\overline{\pi}^m,\overline{R}^m)$ and $\lim_{m\to\infty}(\tilde{x}_m^h,\tilde{\theta}_m^h,\tilde{\varphi}_m^h)=(\tilde{x}^h,\tilde{\theta}^h,\tilde{\varphi}^h)$.

Therefore, since for m large enough, $(\tilde{x}_m^h,\tilde{\theta}_m^h,\tilde{\varphi}_m^h)\in\mathcal{H}_n^h(\overline{p}^m,\overline{\pi}^m,\overline{R}^m)\subset\mathcal{H}_m^h(\overline{p}^m,\overline{\pi}^m,\overline{R}^m)$

$$u^h\left(\tilde{x}_{m,0}^h+\sum_{j\in J}C_j\tilde{\varphi}_{m,j}^h,\left(\tilde{x}_{m,s}^h;s\in S\right)\right)\leq u^h\left(\overline{x}_{m,0}^h+\sum_{j\in J}C_j\overline{\varphi}_{m,j}^h,\left(\overline{x}_{m,s}^h;s\in S\right)\right).$$

Taking the limit as m goes to infinity, we obtain that

$$u^h\left(\tilde{x}_0^h+\sum_{j\in J}C_j\tilde{\varphi}_j^h,(\tilde{x}_s^h;s\in S)\right)\leq u^h\left(\overline{x}_0^h+\sum_{j\in J}C_j\overline{\varphi}_j^h,(\overline{x}_s^h;s\in S)\right),$$

which contradicts the existence of a plan that improve the utility of agent h at $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$. □

It follows from Lemma 6 and the monotonicity of utility function that $(\overline{p}_s;s\in S^*)\gg 0$. Therefore, for any $j\in J$, by the definition of n_k-equilibria and the fact that $\overline{R}_{s,j}^{n_k}$ converges to $\overline{R}_{s,j}$ for each $s\in S$, Assumption (A6) assures that there is a state of nature $s(j)\in S$ such that

$$\overline{R}_{s(j),j}\geq \overline{D}_{s(j),j}(\overline{p}_{s(j)})=\min\left\{\overline{p}_{s(j)}A_{s(j),j},\overline{p}_{s(j)}Y_{s(j)}(C_j)\right\}>0.$$

Furthermore, this last property jointly with the monotonicity of preferences guarantees that, for any $j\in J$, the unitary price $\overline{\pi}_j$ is strictly positive.

Lemma 7. *Suppose that Assumptions (A1)-(A5) hold. Then for each $j\in J$, $\overline{p}_0C_j>\overline{\pi}_j$.*

Proof. Let $h\in\mathbb{P}$. Suppose that there is a $j\in J$ such that, $\overline{p}_0C_j\leq\overline{\pi}_j$. Then, agent h may sell any quantity $a>0$ of debt contract j, to obtain resources at $t=0$ that allow him to consume the bundle $w_0^h+C_ja\gg 0$. This position in the asset j has a limited commitment at any state of nature $s\in S$. In fact, the agent will never pay more than $\Phi_s^h(\overline{p}_s,\overline{R}_s,w_0^h+C_ja,0,ae(j))$, resources that he always have, where $e(j)\in\mathbb{R}^J$ is the canonical vector on j-th coordinate. Therefore, independent of a, he may consume (at least) at any state of nature $s\in S$ the bundle $(1-\lambda_s)w_s^h$ which has strictly positive coordinates as a consequence of Assumptions (A4) and (A5). Using this strategy agent h could improve, for a large enough, his utility function in relation to the level that he obtained with plan $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$. A contradiction. □

Lemma 8. *Suppose that Assumptions (A1), (A2), (A4) and (A5) hold. Then, $\{g_{n_k}\}_{n_k\geq n^*}$ is uniformly integrable and, for each $h\in[0,1]$, $\{g_{n_k}(h)\}_{n_k\geq n^*}$ is bounded.*

Proof. For each $h\in[0,1]$, $\{g_{n_k}(h)\}_{n_k\geq n^*}$ is bounded if the sequence $\left\{(\overline{x}_{n_k}^h,\overline{\theta}_{n_k}^h,\overline{\varphi}_{n_k}^h)\right\}_{n_k\geq n^*}$ is bounded too. Since $(\overline{p},\overline{\pi},(\overline{p}_0C_j-\overline{\pi}_j)_{j\in J})\gg 0$, there exists $\epsilon>0$ and $T^*\in\mathbb{N}$ such that $(\overline{p},\overline{\pi},(\overline{p}_0C_j-\overline{\pi}_j)_{j\in J})\gg\epsilon(1,\ldots,1)$ and, for any $n_k>T^*$,

$$\|\left(\overline{p}^{n_k},\overline{\pi}^{n_k},(\overline{p}_0^{n_k}C_j-\overline{\pi}_j^{n_k})_{j\in J}\right)-(\overline{p},\overline{\pi},(\overline{p}_0C_j-\overline{\pi}_j)_{j\in J})\|_{\max}\leq\epsilon.$$

Therefore, since $\| \left(\overline{p}^{n_k}, \overline{\pi}^{n_k}, (\overline{p}_0^{n_k} C_j - \overline{\pi}_j^{n_k})_{j\in J}\right) \|_{\max} \gg 0$, using individuals' first period budget constraints, we have that, for any $(j,\ell) \in J \times L$ and for each $n_k > T^*$,

$$0 \leq \left(\overline{x}^h_{n_k,0,\ell}, \overline{\theta}^h_{n_k,j}, \overline{\varphi}^h_{n_k,j}\right) \leq \left(\frac{\overline{p}_0^{n_k} w_0^h}{\overline{p}_{0,\ell}^{n_k}}, \frac{\overline{p}_0^{n_k} w_0^h}{\overline{\pi}_j^{n_k}}, \frac{\overline{p}_0^{n_k} w_0^h}{\overline{p}_0^{n_k} C_j - \overline{\pi}_j^{n_k}}\right).$$

In addition, for any $(s,\ell) \in S \times L$,

$$0 \leq \overline{x}^h_{n_k,s,\ell} \leq \frac{\overline{p}_s^{n_k}\left(w_s^h + Y_s(\overline{x}^h_{n_k,0} + \sum_{j\in J} C_j \overline{\varphi}^h_{n_k,j})\right) + \sum_{j\in J} \overline{R}^{n_k}_{s,j} \overline{\theta}^h_{n_k,j}}{\overline{p}^{n_k}_{s,\ell}}.$$

Let $\zeta = \min_{(s,\ell,j)\in S^* \times L \times J} \left\{\overline{p}_{s,\ell},\, \overline{\pi}_j, \overline{p}_0 C_j - \overline{\pi}_j\right\}$ and $\Pi_0 = \frac{1}{\zeta - \epsilon}\|\overline{w}\|_{\max}$ (which is well defined as a consequence of the definition of ϵ). Then, for each $n_k > T^*$,

$$0 \leq \max_{(\ell,j)\in L\times J} \left\{\overline{x}^h_{n_k,0,\ell}, \overline{\theta}^h_{n_k,j}, \overline{\varphi}^h_{n_k,j}\right\} \leq \Pi_0,$$

and for any $s \in S$,

$$0 \leq \max_{\ell\in L} \overline{x}^h_{n_k,s,\ell} \leq \Pi_s := \Pi_0 \left(1 + \frac{1}{\zeta - \epsilon}\left\| Y_s\left((1,\ldots,1) + \sum_{j\in J} C_j\right)\right\|_{\max} + \frac{2\overline{A}}{\zeta - \epsilon}\#J\right).$$

Therefore, for any $h \in [0,1]$, each component of the non-negative sequence $\left\{(\overline{x}^h_{n_k},\, \overline{\theta}^h_{n_k},\, \overline{\varphi}^h_{n_k})\right\}_{n_k \geq n^*}$ is bounded from above by $\Pi := \max_{s\in S^*} \Pi_s$. Since the upper bound of $\{g_{n_k}(h)\}_{n_k \geq n^*}$ is independent of $h \in [0,1]$, the family of functions $\{g_{n_k}\}_{n_k\geq n^*}$ is uniformly integrable (see Hildenbrand (1974, page 52)). □

It follows from Lemma 8 that the sequence of non-negative integrable functions $\{g_{n_k}\}_{n_k\geq n^*}$ satisfies the assumptions of the strong version of the multidimensional Fatou's Lemma (see Hildenbrand (1974, page 69)). Thus, we can found a set $\widehat{\mathbb{P}} \subset [0,1]$ of full measure ($\mu(\widehat{\mathbb{P}}) = 1$) and an integrable function $\widehat{g} : [0,1] \to \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}$, defined by $\widehat{g}(h) := (\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h, (\widehat{\rho}^h_{s,j})_{(s,j)\in S\times J})$ such that, for each agent $h \in \widehat{\mathbb{P}}$ there is a subsequence of $\{g_{n_k}(h)\}_{n_k\geq n^*}$ that converges to $\widehat{g}(h)$, and

$$\int_{[0,1]} \widehat{g}(h)dh = \lim_{k\to\infty} \int_{[0,1]} g_{n_k}(h)dh.^9$$

In addition, the strictly positivity of commodity prices at any state of nature $s \in S$, i.e. $\overline{p}_s \gg 0$, implies that (see footnote 8 above),

$$(\widehat{\rho}^h_{s,j})_{(s,j)\in S\times J} = (\beta_s^h(\overline{p}_s, \overline{R}_s, \widehat{x}_0^h, \widehat{\theta}^h, \widehat{\varphi}^h)\, \widehat{\varphi}^h_j)_{(s,j)\in S\times J}.$$

It follows from condition (3.2) on the definition of n_k-equilibria, taking the limit as k goes to infinity, that there is no excess of demand in physical or in financial markets, i.e.,

[9]Note that, functions g and $\widehat{g}$ which satisfy, respectively, the weak and strong versions of multidimensional Fatou's Lemma do not need to coincide.

$$\int_{[0,1]} \left(\widehat{x}_0^h + \sum_{j\in J} C_j \widehat{\varphi}_j^h - w_0^h \right) dh \leq 0, \qquad \int_{[0,1]} \widehat{\theta}^h dh \leq \int_{[0,1]} \widehat{\varphi}^h dh,$$

$$\int_{[0,1]} \left(\widehat{x}_s^h - w_s^h - Y_s \left(\widehat{x}_0^h + \sum_{j\in J} C_j \widehat{\varphi}_j^h \right) \right) dh \leq 0, \quad \forall s \in S.$$

On the other hand, for any $h \in \widehat{\mathbb{P}}$, identical arguments to those made on Lemma 6 assure that $(\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h)$ is an optimal choice for agent h on the budget set $B^h(\overline{p}, \overline{\pi}, \overline{R})$ and, therefore, budget constraints are satisfied as equality by $(\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h)$. Thus, since $\mu([0,1] \setminus \widehat{\mathbb{P}}) = 0$, integrating over agents, we obtain that,

$$\overline{p}_0 \int_{[0,1]} \left(\widehat{x}_0^h + \sum_{j\in J} C_j \widehat{\varphi}_j^h - w_0^h \right) dh + \sum_{j\in J} \overline{\pi}_j \int_{[0,1]} (\widehat{\theta}_j^h - \widehat{\varphi}_j^h) dh = 0;$$

and for each $s \in S$,

$$\overline{p}_s \int_{[0,1]} \left(\widehat{x}_s^h - w_s^h - Y_s \left(\widehat{x}_0^h + \sum_{j\in J} C_j \widehat{\varphi}_j^h \right) \right) dh = \sum_{j\in J} \overline{R}_{s,j} \int_{[0,1]} \widehat{\theta}_j^h dh - \int_{[0,1]} M_s^h(\overline{p}_s, \overline{R}_s, \widehat{x}_0^h, \widehat{\theta}^h, \widehat{\varphi}^h) dh.$$

Since $(\overline{p}_0, \overline{\pi}) \gg 0$ and there is no excess of demand in physical or in financial markets, it follows that,

$$\int_{[0,1]} \left(\widehat{x}_0^h + \sum_{j\in J} C_j \widehat{\varphi}_j^h - w_0^h \right) dh = 0, \qquad \int_{[0,1]} (\widehat{\theta}^h - \widehat{\varphi}^h) dh = 0.$$

Using condition (3.3) of the definition of n_k-equilibria and taking the limit as k goes to infinity, it follows that, for any $(s,j) \in S \times J$,

$$\overline{R}_{s,j} \int_{[0,1]} \widehat{\theta}_j^h dh = \int_{[0,1]} D_{s,j}(\overline{p}_s) \widehat{\varphi}_j^h dh + \int_{[0,1]} [\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)]^+ \beta_s^h(\overline{p}_s, \overline{R}_s, \widehat{x}_0^h, \widehat{\theta}^h, \widehat{\varphi}^h)\, \widehat{\varphi}_j^h dh.$$

Adding on $j \in J$, for a fixed $s \in S$, we obtain that

$$\sum_{j\in J} \overline{R}_{s,j} \int_{[0,1]} \widehat{\theta}_j^h dh = \int_{[0,1]} M_s^h(\overline{p}_s, \overline{R}_s, \widehat{x}_0^h, \widehat{\theta}^h, \widehat{\varphi}^h) dh.$$

As $\overline{p}_s \gg 0$, for any $s \in S$,

$$\int_{[0,1]} \left(\widehat{x}_s^h - w_s^h - Y_s \left(\widehat{x}_0^h + \sum_{j\in J} C_j \widehat{\varphi}_j^h \right) \right) dh = 0.$$

Therefore, market clearing condition holds for the allocation $((\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h); h \in [0,1])$. Moreover, as we said above, for any $h \in \widehat{\mathbb{P}}$, $(\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h)$ is an optimal allocation in $B^h(\overline{p}, \overline{\pi}, \overline{R})$.

Since $((\overline{p}_s)_{s\in S^*}, \overline{\pi}, (\overline{p}_0 C_j - \overline{\pi}_j)_{j\in J}) \gg 0$, each agent $h \in [0,1]$ has a compact budget set $B^h(\overline{p}, \overline{\pi}, \overline{R})$. Continuity of utility functions (Assumption (A1)) assures that any agent $h \in [0,1] \setminus \widehat{\mathbb{P}}$ has an optimal allocation $(\breve{x}^h, \breve{\theta}^h, \breve{\varphi}^h) \in B^h(\overline{p}, \overline{\pi}, \overline{R})$. Thus, if we give to h the allocation $(\breve{x}^h, \breve{\theta}^h, \breve{\varphi}^h)$ instead of

$(\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h)$, we assure that all consumer maximize their utility function without change the validity of markets clearing condition (because $[0,1] \setminus \widehat{\mathbb{P}}$ has zero measure).

Therefore,

$$\Big((\overline{p}, \overline{\pi}, \overline{R}); ((\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h); h \in \widehat{\mathbb{P}}); ((\breve{x}^h, \breve{\theta}^h, \breve{\varphi}^h); h \in [0,1] \setminus \widehat{\mathbb{P}})\Big)$$

is an equilibrium of $\mathcal{E}$. This concludes the proof of equilibrium existence in our economy.

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