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WALRASIAN PRICES IN A MARKET WITH CONSUMPTION RIGHTS

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Walrasian prices in a market with consumption rights *

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Abstract

In this paper we consider an exchange economy where there is an external restriction for the consumption of goods. This restriction is defined by both a cap on consumption of certain commodities and the requirement of an amount of rights for the consumption of these commodities. The caps for consumption are imposed exogenously due to the negative effects that the consumption may produce. The consumption rights are distributed among the agents. This fact leads to the possibility of establishing licence or consumption rights markets. These consumption rights do not participate in agents' preferences, however the individual's budgetary constraint may be modified, leading to a reassignment of resources.

The aim of this paper is to show the existence of a Walrasian equilibrium price system linking tradable rights prices with commodity prices.

Keywords: competitive equilibrium, quotas, consumption rights, cap-and-trade program.

JEL Classification: D51, D62, Q52.

1 Introduction

Tradable-licence systems are the focus of current interest in market-based natural resources or environmental policies. For example, a system of licences is interesting as it could provide a mean to achieve efficient solutions to set restrictions on fishing for certain fish species or in order to organize a market of emission licences or pollution rights in a decentralized manner.

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An emission licence system confers the agents holding it the right to consume. In the examples above, the right to capture a protected species of fish or to emit pollutants at a certain rate. However, it is not always desirable to allow such rights to be transferred on a one-to-one basis. In a market system these licences should be tradable and the desirable rule governing exchange of rights should be based on a market-price system.

These types of models have been analyzed by several authors since mid last century (see Baumol et al. [1] and Montero [7] as general references). Focusing on pricing rights in a competitive basis, a precise formulation of emission licences model appears in the seminal paper by Montgomery [8]. In a scenario where an exchange of such licences between polluters at different locations is considered, Montgomery shows that market equilibrium in emission licences exists and that with some restrictions on the initial allocation of licences, the market equilibrium with emission licences is efficient. On the other hand, Boyd and Conley [3] were the first to treat directly the efficiency problem in presence of externalities opposed to an indirectly way through *Arrovian* commodities, arguing that the essential non-convexities highlighted by Starrett [9] are due to unboundedness of the negative effects rather than the externalities itself.

Later, Conley and Smith [4] extended the Boyd and Conley model to allow firms to benefit from public goods and be damaged by externalities, proving the existence of a competitive equilibrium and stating a first welfare theorem. Their main result could be viewed as a type of general equilibrium Coase theorem. More recently, the paper by A. Mandel [6], focuses on the influence on the general equilibrium of an economy of the opening of markets of allowances. Assuming there existed an equilibrium before the opening of allowances markets, this paper describes the changes in the firms behavior which guarantee that an equilibrium can be reached in the enlarged economy.

The presence of an Arrovian market with consumable (and tradable) licences or the models considered by Boyd and Conley, Conley and Smith, or Mandel imply to re-consider the pollutants as crucial consumptions goods as well as key input factors for production, which drive us to the necessity of re-defining the individual preferences and production sets in order to take into account these new factors in their formulations. How to model changes in preferences (and production sets) since to the presence of new goods in the market is certainly an open question, for which we do not know a satisfactory answer at this moment.

In this paper, we consider a scenario in which limits to the consumption of certain commodities have been established exogenously and that the consumption of these commodities requires the availability of certain amount of rights or licences for its consumption. The scenario may reflect a situation where, due to binding international agreements, limits to the consumption of certain raw materials or limits to the capture of protected species have been established in order to restrict the potential negative effects produced by their consumption. These negative effects can be, for example, the extinction of a species or different types of environmental pollution.

Our aim is to set a simple model in order to analyze the immediate consequences of setting a limit on the consumption of certain commodities and, at the same time, a market of licences required for the consumption. For it, we consider an exchange economy with externalities (the individual's preferences depend on private consumption goods chosen by this individual and on the entire consumption plan chosen by other agents in the economy). Consumption rights do not participate in preferences but may modify the budgetary constraints.

The restrictions of the model primarily affect the agents' consumption sets. Agents may not consume certain quantities of specific commodities even when these form part of their endowments. Secondly, it may affect the agents' budget sets, since in order to consume they will need to have the required rights. If an agent does not have those rights, she may buy them investing part of her income coming from consumption goods, or on the contrary, if she has any licences left over, she could sell them to get an additional income.

It is also assumed that the estimated negative effects, and consequently the rights required for the consumption of specific commodities, could depend not only on the quantity of those commodities but also on the entire consumption plan selected by the consumer. Our objective here is to reflect the situation in which a consumption plan, which entails a high technology, may involve less adverse effects, and consequently to require fewer consumption rights than another less technological consumption plan.

This model assumes that each agent is endowed with a certain amount of each type of consumption rights which are necessary for consumption and that can be traded. The agent's choice of a specific consumption plan requires that she has the inherent licences for that consumption.

Our approach differs from other previous works in several aspects. Firstly, we do not consider explicitly production. In our model, agents evaluate their utility considering all the consequences involved in their consumption plan. Thus, our model is a pure exchange market in which the consumption rights are traded at the same time as the commodities, that is, consumption rights must be required at the same time as the buying contracts for raw materials are signed, no matter what the raw materials are used for. Therefore, more importantly, we do not require to measure the actual negative effects of consumption. Instead, we only suppose the existence of an external function which evaluates the potential negative effects derived from each contract, by mapping every consumption plan (contract) into a theoretical amount of rights of each type. Secondly, we do not need to introduce any other *type of good* in individual' preferences and neither in the production sector (*Arrovian markets*), which avoid us to justify how preferences and/or production sets could be distorted by introducing them.

Due to the presence of externalities in consumption (as we setup the model in Section 2), we introduce the concept of *Nash - Walras equilibria* as a competitive outcome in our framework. This concept coincides with the standard Walras notion if we do not consider externalities.

In Section 3 we prove a Walras' Law for our equilibrium concept. The main result of this paper is Theorem 1 in Section 4, which establish the existence of a Nash-Walras equilibrium under general conditions on the fundamentals of the economy. Additionally, in Section 5 we present an efficiency property of our equilibrium, Theorem 2. Efficiency is defined here as *Nash - Pareto optimality*, which, without externalities coincides with the standard concept of strong Paretianity. Finally, Section 6 is devoted to the conclusion remarks and further developments.

2 The model

Following the standard Arrow-Debreu model, let us consider an economy with $m \in \mathbb{N}$ consumers and $\ell \in \mathbb{N}$ different consumption goods; each consumer $i \in I = \{1, 2, \ldots, m\}$ is endowed with consumption goods denoted by $\omega_i \in \mathbb{R}^\ell_+$ and the corresponding consumption set is $X_i \subseteq \mathbb{R}^\ell_+$. Following, we set $\omega = \sum_{i \in I} \omega_i$, $X = \prod_{i \in I} X_i$ and given $i \in I$, we define

$$X_{-i} = \prod_{j \in I \setminus \{i\}} X_j.$$

In order to incorporate externalities in consumption, preferences of an individual $i \in I$ will be represented by a utility function

$$u_i : X_{-i} \times X_i \to \mathbb{R}.$$

We assume that limits to the consumption of certain commodities have been established exogenously due to binding international agreements established, where consumption of these commodities requires the availability of certain rights or licences. After an exogenous *Cap-setting Process*, limits to consumption are given by the mapping

$$f : \mathbb{R}^\ell_+ \to \mathbb{R}^k_+,$$

which defines the amount of the each type of $k \in \mathbb{N}$ negative effects that could produce the consumption of the allocation $x \in \mathbb{R}^\ell_+$.

For $j \in K = \{1, \ldots, k\}$, the *Cap-setting Process* sets a limit $R_j \in \mathbb{R}_{++}$ on the total allocation of the economy; we set

$$R = (R_j) \in \mathbb{R}^k_{++}.$$

In our model, the cap-setting process yet mentioned implies that for each $j \in K$, any consumption plan $x_i \in X_i$, $i \in I$, should comply with

$$\sum_{i \in I} f_j(x_i) \leq R_j,$$

where f_j denotes the $j \in K$ component of f that defines the caps to consumption that have been exogenously establish. Observe that $f_j(x_i)$ could be the amount of commodity j representing certain raw material for which a cap has been established in order to restrict the potential negative effects that this consumption will produce. However, we are considering a more general setting; in this model, each one of the potential negative effects and, consequently each cap, is measured globally in the sense that it depends not only on the amount of a given commodity but on the global consumption plan of the individuals. Proceeding in this way, we have in mind, for example, that a more technological consumption plan may produce less negative effects than a technologically poorer alternative.

On the other hand, we assume that for each $j \in K$ there is a type of *consumption right* and that each individual $i \in I$ is endowed with an amount of each of them. Formally, each agent $i \in I$ is endowed with a vector

$$r_i = (r_i^j) \in I\!R_+^k$$

in such a way that

$$\sum_{i \in I} r_i^j = R_j, \; j \in K.$$

If agent $i \in I$ decides to consume $x \in X_i$ then she must have an *amount* $f(x) \in I\!R_+^k$ of each consumption rights. One key assumption in our model is that the consumption rights can be traded in the market and that they do not participate in the individual's preferences. The fact that consumption rights can be traded in the market implies that any individual may exchange them with consequences on the size of her budgetary set; as for consumption goods, prices for consumption rights will be determined endogenously as part of the equilibrium.

Thus, the difference

$$r_i - f(x) \in I\!R^k$$

defines either the amount of consumption rights that individual $i \in I$ may sell in the market (those for which the corresponding component is positive) or those she needs to buy since his initial endowment of the corresponding consumption right is not enough to support the consumption of x (negative components).

If the price for consumption rights is $s \in I\!R_+^k$, then the consumption of x as before implies that the total wealth he can obtain (or pay if negative) from this side is

$$s \cdot [r_i - f(x)] \in I\!R.$$

In the following, Δ will be the Simplex in $I\!R^{\ell+k}$ and for $n \in I\!N_+$ and $x, \; y \in I\!R^n$, we say that $x \leq_n y$ iff $x_i \leq y_i$, for each $i = 1, 2, \ldots, n$, $x <_n y$ iff $x \leq_n y$ and $x \neq y$ and, $x <<_n y$ iff $x_i < y_i$, for each $i = 1, 2, \ldots, n$. Finally, 0_n is the zero in $I\!R^n$.

Definition 1 *For $(p, s) \in \Delta$, the budgetary set for individual $i \in I$ at prices (p, s) is defined by*

$$B_i(p, s) = \left\{ \xi_i \in X_i \mid p \cdot \xi_i \leq p \cdot \omega_i + s \cdot [r_i - f(\xi_i)] \right\}.$$

Definition 2 *An economy with consumption rights and externalities is defined as*

$$\mathcal{E}_R = (X_i, (u_i), (\omega_i), (r_i), f)_{i \in I}.$$

The corresponding economy without consumption rights ("exchange economy with externalities") will be denoted by

$$\mathcal{E} = (X_i, (u_i), (\omega_i))_{i \in I}.$$

In order to define the equilibrium notion for economy $\mathcal{E}_R$, we should consider feasibility in both consumption goods and consumption rights.

Definition 3 *We say that $x = (x_i) \in X$ is a feasible allocation for economy $\mathcal{E}_R$ if*

$$\sum_{i \in I} x_i \leq_\ell \omega \in I\!R^\ell_+$$

and

$$\sum_{i \in I} f(x_i) \leq_k R \in I\!R^k_+.$$

The set of feasible allocation for economy $\mathcal{E}_R$ is denoted by $\mathcal{F}_R$.

Remark 1 Observe that the endowments $(\omega_i) \in X$ need not to be a feasible allocation for economy $\mathcal{E}_R$. This is the case if for some j

$$\sum_{i \in I} f_j(\omega_i) > R_j.$$

More in general, for $j \in K$ suppose that f_j is a convex function and $f_j(0_\ell) = 0$; if $x = (x_1, x_2, ..., x_m) \in X$ allocates the total endowments, that is, $\sum_{i \in I} x_i = \omega$, then we have that

$$f_j(\omega/m) \leq \frac{1}{m} \sum_{i \in I} f_j(x_i) \leq \frac{1}{m} R_j.$$

Consequently, if $R_j < m f_j(\omega/m)$ the cap is effective. That is, it is not possible to allocate the total endowments of the economy.

Finally, the definition below is a natural extension of the competitive equilibrium notion we have for an exchange economy[1].

Definition 4 *We say that $((p^*, s^*), (x_i^*)) \in \Delta \times I\!R^{m\ell}_+$ is a Nash-Walras equilibrium for economy $\mathcal{E}_R$ if*

(a) $x^* = (x_i^*) \in \mathcal{F}_R$

(b) for each $i \in I$, $x_i^* \in B_i(p^*, s^*)$, and x_i^* maximizes $u_i(x^*_{-i}, \cdot)$ on $B_i(p^*, s^*)$.

3 Walras' Law and some direct consequences

We begin this Section with the following straightforward lemmata, which will be useful to show the Walras' Law in our context (Proposition 1) and a version of the First Welfare Theorem we present in Section 5.

[1]In the following, for $x = (x_i) \in X$, we adopt the notation $u_i(x) = u_i(x_{-i}, x_i)$.

Lemma 1 *Suppose that* $f : I\!R^\ell_+ \to I\!R^k_+$ *is continuos and that for* $i \in I$ *and for any* $x_{-i} \in X_{-i}$, $u_i(x_{-i}, \cdot) : X_i \to I\!R$ *is locally non-satiated*[2]. *Given* $((p^*, s^*), (x_i^*))$ *a Nash-Walras equilibrium of* $\mathcal{E}_R$, *if for* $x_i \in X_i$ *holds that* $u_i(x^*) \leq u_i(x_{-i}^*, x_i)$, *then*

$$p^* \cdot x_i \geq p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i)] .$$

Proof. Suppose that $p^* \cdot x_i < p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i)]$. Since f is continuous, there exist $\epsilon > 0$ such that

$$p^* \cdot x_i' < p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i')]$$

for all $x_i' \in B(x_i, \epsilon)$. Therefore, by local non satiation, there are a point $z \in B(x_i, \epsilon)$ such that $u_i(x_{-i}^*, x_i) < u_i(x_{-i}^*, z)$ and then $u_i(x^*) < u_i(x_{-i}^*, z)$, which contradicts that (x_i^*) is the equilibrium allocation at prices (p^*, s^*). E.O.P

A direct consequence of Lemma 1 is the following proposition.

Proposition 1 Walras' Law

Under the conditions of Lemma 1, if $((p^*, s^*), (x_i^*))$ *is a Nash-Walras equilibrium of* $\mathcal{E}_R$ *then*

$$p^* \cdot \left[\sum_{i \in I} x_i^* - \omega \right] = 0, \quad s^* \cdot \left[\sum_{i \in I} f(x_i^*) - R \right] = 0.$$

Proof. From Lemma 1, for each $i \in I$, $p^* \cdot x_i^* = p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i^*)]$, which lead us to conclude

$$p^* \cdot \left[\sum_{i \in I} x_i^* - \omega \right] + s^* \cdot \left[\sum_{i \in I} f(x_i^*) - R \right] = 0. \tag{1}$$

Since $\sum_{i \in I} x_i^* \leq_\ell \omega$, $\sum_{i \in I} f(x_i^*) \leq_k R$ and $(p^*, s^*) \in I\!R^{\ell+k}_+$, follows that $p^* \cdot \left[\sum_{i \in I} x_i^* - \omega \right] \leq 0$ and $s^* \cdot \left[\sum_{i \in I} f(x_i^*) - R \right] \leq 0$, which along with (1) implies the desired result. E.O.P

Remark 2 For a Nash - Walras equilibrium $((p^*, s^*), (x_i^*))$, the fact that consumption rights may effectively restrict the consumption of a good $k \in \{1, 2, \ldots, \ell\}$ corresponds to say that $\sum_{i \in I} x_{ik}^* - \omega_{ik} < 0$; under this situation, the Walras' Law implies that $p_k^* = 0$. Note that this fact does not depend on the distribution of consumption rights among individuals but only depends on the aggregate value of the consumption rights. In this situation, as we will see in the next example, the individual's assignment of consumption rights could have consequences on their welfare in the equilibrium, allowing further analysis regarding public policy through the assignment of consumption rights among agents. This will not be treated in this work.

[2]That is, for any $x_{-i} \in X_{-i}$, $\epsilon > 0$ and $x_i \in X_i$, there exists $x_i' \in B(x_i, \epsilon) \cap X_i$ such that $u_i(x_{-i}, x_i) < u_i(x_{-i}, x_i')$, where $B(x_i, \epsilon)$ is the open ball with center x_i and radius ϵ.

Suppose that consumption rights effectively restrict the consumption of a good $k \in \{1, 2, \dots, \ell\}$ and that for some consumer i the good k is desirable, that is, for any positive λ, $u(x_i^* + \lambda e_k) > u(x_i^*)$, where e_k is the k-th vector of the canonic basis of $I\!R^\ell$. From this, follows immediately that $p_k^* = 0$ and, from the the budgetary constrain, we have that for some $j \in K$, $s_j^* > 0$ and $f_j(x_i^* + \lambda e_k) > f_j(x_i^*)$. In consequence, an effective cap on a commodity implies that the equilibrium price of that commodity is zero and that the price of the consumption right becomes the relevant price.

On the contrary, note that when the level of consumption rights is large enough, the price of the consumption rights becomes zero at the equilibrium and the economy is equivalent to a classical exchange market with externalities $\mathcal{E}$.

Example 1 In order to define economy $\mathcal{E}_R$, suppose $m = 2$, $\ell = 2$ and that individual's preferences are given by $u_1(x_1, x_2) = u_2(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, with $0 < \alpha < 1$. Endowments of goods are $(\omega_{i2}, \omega_{i2}) \in I\!R^2_+$, $i = 1, 2$; set $\omega_j = \omega_{1j} + \omega_{2j} > 0$, $j = 1, 2$. Suppose additionally that $K = 1$, $f(x_1, x_2) = bx_2$ (with $b > 0$) and endowment for consumption rights are $r_i \geq 0$, $i = 1, 2$. Set $R = r_1 + r_2 > 0$. The economy $\mathcal{E}$ is defined by u_i and $(\omega_{i1}, \omega_{i2})$, $i = 1, 2$, as before. In the following, assume that good one is the *numerary* and prices for good two and consumption rights are denoted by p and s respectively. From the monotonicity of the involving functions, the consumer's problem for agent $i = 1, 2$ is

$$\max_{x_{i1}, x_{i2}} x_{i1}^\alpha x_{i2}^{1-\alpha} \quad s.t. \quad x_{i1} + p x_{i2} = \omega_{i1} + p\omega_{i2} + s\left[r_i - b x_{i2}\right], \quad x_{i1},\ x_{i2} \geq 0,$$

whose unique solution is

$$x_{i1}(p, s) = \alpha\left[\omega_{i1} + p\omega_{i2} + s r_i\right], \quad x_{i2}(p, s) = (1-\alpha)\left[\frac{\omega_{i1} + p\omega_{i2} + s r_i}{p + bs}\right], \ i = 1, 2.$$

The equilibriums conditions for goods one and two are, respectively,

$$x_{11}(p, s) + x_{21}(p, s) = \omega_1 \quad \Leftrightarrow \quad \alpha\left[\omega_1 + p\omega_2 + sR\right] = \omega_1 \tag{2}$$

$$x_{12}(p, s) + x_{22}(p, s) \leq \omega_2 \quad \Leftrightarrow \quad (1-\alpha)\left[\frac{\omega_1 + p\omega_2 + sR}{p + bs}\right] \leq \omega_2. \tag{3}$$

Combining (2), (3) and the budget constrain, for any $s \geq 0$

$$s\left[R - b\omega_2\right] \leq 0. \tag{4}$$

For the case $R > b\omega_2$, the unique equilibrium price is

$$s^c = 0, \quad p^c = \left(\frac{1-\alpha}{\alpha}\right)\frac{\omega_1}{\omega_2},$$

which coincides with the equilibrium price for the economy $\mathcal{E}$. For the case $R = b\omega_2$, there are infinitely many equilibrium prices $(p,s) \in I\!R^2_+$, parametrized by the relation $p + sb = p^c$.

For the case $R < b\omega_2$, from (4) we have that $s \geq 0$. However, note that $s = 0$ is not an admissible solution, since in such case the aggregated equilibrium demand for consumption goods two would be equal to those obtained for economy $\mathcal{E}$ (i.e ω_2), which, by condition, is not a feasible allocation from the consumption rights side. In consequence, we may assume $s > 0$ and then, in order to preserve feasibility from the consumption rights side, from (3) holds that

$$b(1-\alpha)\left[\frac{\omega_1 + p\omega_2 + sR}{p + bs}\right] \leq R.$$

If we denote by R' the consumption effectively employed by agents, follows that

$$b(1-\alpha)\omega_1 + p[b(1-\alpha)\omega_2 - R'] + s[b(1-\alpha)R - bR'] = 0, \tag{5}$$

from which, along with (2) we conclude that

$$p\left[\omega_2 - \frac{R'}{b}\right] + s[R - R'] = 0.$$

Since $R' \leq R < b\omega_2$, in order to obtain positive equilibrium prices we must impose $R' = R$, which lead us to conclude that the equilibrium price for good two is $p^* = 0$, and from (2) the equilibrium price for consumption rights should be

$$s^* = \frac{(1-\alpha)\omega_1}{\alpha R}.$$

Regarding good one, the equilibrium allocations are

$$x^r_{i1} = \alpha\left[\omega_{i1} + \frac{(1-\alpha)\omega_1}{\alpha R} r_i\right] = \alpha\omega_{i1} + (1-\alpha)\omega_1\frac{r_i}{R}, \;\; i = 1,2, \tag{6}$$

which, for individual $i = 1,2$, would be greater than those obtained in the exchange economy without consumption rights provided that

$$\frac{r_i}{R} > \frac{\omega_{i2}}{\omega_2}.$$

Regarding good two, given $\delta = R - b\omega_2 > 0$, the aggregated demand at the equilibrium is given by

$$(1-\alpha)\left[\frac{\omega_1 + sR}{bs}\right] = \omega_2 - \frac{\delta}{b} < \omega_2. \tag{7}$$

Note that $R < b\omega_2$ implies that for some $i = 1, 2$, $r_i < b\omega_{i2}$. Thus, the initial endowments of goods and rights do not necessarily belong to the budgetary set for this individual, at any price. This fact is relevant in our model since it implies that we cannot use standard arguments to prove the existence of equilibrium in our setting by considering an extended economy where consumption rights appear as new commodities in the market, even though they do not directly participate in agent's preferences.

Finally, from (6), the presence of consumption rights in the market imply a redistribution of good one between agents that otherwise can not be reached as a competitive outcome in economy $\mathcal{E}$, unless a redistribution of endowments is carried out. However, from (7) we also have that the presence of them may effectively restrict the consumption of goods, implying an *excess of supply* that cannot be assigned to any individual. Thus, the consumption rights cannot necessarily be interpreted as a tax mechanism whose role is to reach a certain point in the contract curve of economy $\mathcal{E}$.

4 Existence of equilibrium

For the existence of equilibrium in our model we will consider quite standard hypotheses on the fundamentals of the economy. The strongest condition we are assuming for the existence of equilibrium result is **SS** (*strong survival* condition).

Assumption C. For each $i \in I$, $X_i \subseteq \mathbb{R}^\ell_+$ is convex, closed and $0_\ell,\ \omega_i \in X_i$.

Assumption SS. For each $i \in I$, $\omega_i \in \mathbb{R}^\ell_{++}$ and $r_i \in \mathbb{R}^k_{++}$.

Assumption R. For each $j \in K$, $f_j : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ is convex, continuous and $f_j(0_\ell) = 0$ (i.e., $f(0_\ell) = 0_k$).

Assumption U. For each $i \in I$, $u_i : X \to \mathbb{R}$ is continuous and for each $x_{-i} \in X_{-i}$, $u_i(x_{-i}, \cdot) : X_i \to \mathbb{R}$ is locally non-satiated and quasi-concave.

In order to facilitate the demonstration of our main result, we introduce the auxiliary economy $\mathcal{E}_R^M$, which differs from $\mathcal{E}_R$ only in the consumption sets that now, for individual $i \in I$, is defined by[3]

[3]The closure of $A \subseteq \mathbb{R}^n$ is denoted by clA and the Euclidean norm of $x \in \mathbb{R}^n$ by $\|x\|$.

$$X_i^M = X_i \cap clB\left(0_\ell, M\|\omega\|\right),$$

with $M > 1$ a given constant[4]. We set $X^M = \prod_{i\in I} X_i^M$ and for $i \in I$, define

$$X_{-i}^M = \prod_{j\in I\setminus\{i\}} X_j^M.$$

Lemma 2 *Under Assumptions* **C**, **SS** *and* **R**, *for $i \in I$ the correspondence*

$$B_i^M : \Delta \to X_i^M \mid B_i^M(p,s) = \left\{\xi_i \in X_i^M \mid p\cdot\xi_i \le p\cdot\omega_i + s\cdot[r_i - f(\xi_i)]\right\}$$

is continuous.

Proof. From Assumption **C**, it follows directly that for each $i \in I$, B_i^M is a closed correspondence. Since X_i^M is compact it is upper semi-continuous.

Now, in order to show the lower semi-continuity of B_i^M at any point $(p_0, s_0) \in \Delta$, let G be any open set such that $B_i^M(p_0,s_0)\bigcap G \ne \emptyset$ and let ξ belonging to this set. Observe that by Assumption **S** we have that

$$0 < p_0\cdot\omega_i + s_0\cdot[r_i - f(0_\ell)],$$

and therefore, from the convexity of f we conclude that for all $\lambda \in [0, 1)$

$$p_0\cdot\lambda\xi < p_0\cdot\omega_i + s_0\cdot[r_i - f(\lambda\xi)].$$

Let be $\lambda_0 < 1$ such that $\lambda_0\xi \in G$. Since f is continuous, there exists $\epsilon > 0$ such that $\|(p,s) - (p_0,s_0)\| < \epsilon$ implies that

$$p\cdot\lambda_0\xi < p\cdot\omega_i + s\cdot[r_i - f(\lambda_0\xi)],$$

from which we deduce that $B_i^M(p,s)\bigcap G \ne \emptyset$ for all $(p,s) \in \Delta$ such that $\|(p,s) - (p_0,s_0)\| < \epsilon$. This last assertion finally lead us to conclude that B_i^M is a continuous correspondence as required. E.O.P

Theorem 1 Existence of Equilibrium

Under assumptions **C**, **SS**, **R** *and* **U** *there exist a Nash-Walras equilibrium for economy $\mathcal{E}_R$.*

Proof. For $i \in I$ define the function

$$u_i^* : \Delta\times X^M\times X_i^M \to I\!R \mid u_i^*((p,s),x,z) = u_i(x_{-i},z),$$

and the correspondence

$$\mathbf{B}_i^M : \Delta\times X^M \to X_i^M \mid \mathbf{B}_i^M((p,s),x) = B_i^M(p,s).$$

[4]Note that from feasibility condition for consumption bundles, any relevant consumption plan x_i for an individual $i \in I$ should comply with $0_\ell \le x_i \le_\ell \omega$ and therefore $\|x_i\| \le \|\omega\|$.

Note that under assumption **U**, the demand correspondence of the auxiliary economy $\mathcal{E}_R^M$, D_i^M, defined by

$$D_i^M : \Delta \times X^M \to X_i^M \mid$$
$$D_i^M((p,s),x) = \{\xi_i \in \mathbf{B}_i^M((p,s),x) \mid u_i(x_{-i},\xi_i) \geq u_i(x_{-i},z),\ \forall z \in \mathbf{B}_i^M((p,s),x)\},$$

is compact and convex valued and from Lemma 2 and the Maximum Theorem (Berge [2]), it is upper semi-continuous.

Now, following standard approach, for the *additional agent (the market)*, define the function

$$u_0^* : \Delta \times X^M \times \Delta \to I\!R \ \mid$$

$$u_0^*((p,s),x,(p',s')) = p' \cdot \left(\sum_{i\in I} x_i - \omega\right) + s' \cdot \left(\sum_{i\in I} f(x_i) - R\right),$$

and the constant correspondence

$$B_0^M : \Delta \times X^M \times \Delta \to \Delta \ \mid \ B_0^M((p,s),x,(p',s')) = \Delta.$$

The demand of the *market* is defined by the correspondence,

$$D_0^M : \Delta \times X^M \to \Delta \mid D_0^M((p,s),x) =$$
$$\left\{(p',s') \in \Delta \mid (p'-\tilde{p}) \cdot \left(\sum_{i\in I} x_i - \omega\right) + (s'-\tilde{s}) \cdot \left(\sum_{i\in I} f(x_i) - R\right) \geq 0, \forall (\tilde{p},\tilde{s}) \in \Delta\right\},$$

which is convex and compact valued and, again by the Maximum Theorem (Berge, [2]), it is upper semi-continuous.

Thus, if we define

$$D^M : \Delta \times X^M \to \Delta \times X^M \mid D^M = \prod_{i=0}^{m} D_i^M,$$

follows immediately that D^M is compact and convex valued and upper semi continuous and since $\Delta \times X^M$ is convex and compact, from Kakutani's Fixed Point Theorem we conclude that there exist $((p^*,s^*),(x_i^*)) \in \Delta \times X^M$ such that

$$((p^*,s^*),(x_i^*)) \in D^M(((p^*,s^*),(x_i^*))),$$

that is,

(*i*) for each $i \in I$, $x_i^* \in D_i^M((p^*,s^*),(x_i^*))$,

(*ii*) $(p^*,s^*) \in D_0^M((p^*,s^*),(x_i^*))$.

From condition (i), x_i^* maximizes $u_i(x_{-i}^*, \cdot)$ on the budget set $\mathbf{B}_i^M((p^*, s^*), x^*)$. On the other hand, since $x_i^* \in \mathbf{B}_i^M((p^*, s^*), x^*)$, $i \in I$, we have that

$$p^* \cdot x_i^* \leq p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i^*)],$$

which, by summing on all agents, lead us to conclude

$$p^* \cdot \left(\sum_{i \in I} x_i^* - \omega \right) + s^* \cdot \left(\sum_{i \in I} f(x_i^*) - R \right) \leq 0. \tag{8}$$

From condition (ii) and inequality (8), holds that for each $(p, s) \in \Delta$

$$p \cdot \left(\sum_{i \in I} x_i^* - \omega \right) + s \cdot \left(\sum_{i \in I} f(x_i^*) - R \right) \leq 0.$$

Taking $s = 0_k$ and letting p be each vector the canonic basis of $I\!R^\ell$, last inequality implies that

$$\sum_{i \in I} x_i^* - \omega \leq_\ell 0_\ell.$$

In the same way, taking $p = 0_\ell$ and letting s be each vector of the canonic basis of $I\!R^k$, we conclude that

$$\sum_{i \in I} f(x_i^*) - R \leq_k 0_k.$$

Thus, all the foregoing implies that $((p^*, s^*), (x_i^*)) \in \Delta \times X^M$ is an equilibrium for economy $\mathcal{E}_r^M$. In order to show that $((p^*, s^*), (x_i^*))$ is also an equilibrium for economy $\mathcal{E}_R$, let us suppose that for some $i \in I$ there exists $\tilde{x}_i \in X_i \setminus X_i^M$ such that

(a) $u_i(x_{-i}^*, \tilde{x}_i) > u_i(x_{-i}^*, x_i^*)$,

(b) $p^* \cdot \tilde{x}_i \leq p^* \cdot \omega_i + s^* \cdot [r_i - f(\tilde{x}_i)]$.

Taking $\tilde{\lambda} \in]0, 1[$ close enough to one, Assumption **C** implies that $\tilde{\lambda} \tilde{x}_i \in X_i$ and from Assumption **U**, $u_i(x_{-i}^*, \tilde{\lambda} \tilde{x}_i) > u_i(x_{-i}^*, x_i^*)$. Moreover, condition (b) above directly implies

$$p^* \cdot (\tilde{\lambda} \tilde{x}_i) < p^* \cdot \omega_i + s^* \cdot [r_i - f(\tilde{x}_i)]. \tag{9}$$

Additionally, from Assumption **R** is easy to check that $-f(\tilde{x}_i) \leq_k -f(\tilde{\lambda} \tilde{x}_i)$, and then, considering that $s^* \in I\!R_+^k$, inequality (9) finally implies

$$p^* \cdot (\tilde{\lambda} \tilde{x}_i) < p^* \cdot \omega_i + s^* \cdot \left[r_i - f(\tilde{\lambda} \tilde{x}_i) \right]. \tag{10}$$

For $\mu \in]0, 1[$ define

$$x_i^\mu = \mu x_i^* + (1-\mu)\tilde{\lambda}\tilde{x}_i.$$

From (10) and Assumption **R**, holds that $p^* \cdot x_i^\mu < p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i^\mu)]$, and from the quasi-concavity of $u_i(x_i^*, \cdot)$, $u_i(x_{-i}^*, x_i^\mu) \geq u_i(x_{-i}^*, x_i^*)$.

Note now that for $\mu \in]0,1[$ close enough to one, x_i^μ belongs to X_i^M and therefore, from Assumption **U** we have that for some $\epsilon > 0$ there exists $\bar{x}_i \in X_i^M \cap B(x_i^\mu, \epsilon)$ such that

$$u_i(x_{-i}^*, \bar{x}_i) > u_i(x_{-i}^*, x_i^\mu) \geq u_i(x_{-i}^*, x_i^*).$$

Finally, choosing μ sufficiently close to 1, the continuity of f implies that

$$p^* \cdot \bar{x}_i \leq p^* \cdot \omega_i + s^* \cdot [r_i - f(\bar{x}_i)],$$

which contradicts the fact that x_i^* maximizes $u_i(x_{-i}^*, \cdot)$ on $\mathbf{B}_i^M((p^*, s^*), x^*)$. E.O.P

5 Optimality

In order to show a kind of First Welfare Theorem (FWT) in our model, we will consider a particular definition of optimality since, as it is well known in the literature, in presence of externalities there are simple counterexamples where this theorem fails.

The following definition is a particular extension of the Pareto optimum notion for the case of an exchange economy with externalities.

Definition 5 *We say that $x^* = (x_i^*) \in \mathcal{F}_R$ is a Nash-Pareto optimum for the economy $\mathcal{E}_R$ if there does not exist another feasible allocation $(x_i') \in \mathcal{F}_R$ such that for each $i \in I$, $u_i(x^*) \leq u_i(x_{-i}^*, x_i')$ and for some $i_0 \in I$, $u_{i_0}(x^*) < u_{i_0}(x_{-i_0}^*, x_{i_0}')$.*

Theorem 2 First Welfare Theorem

Suppose that the economy $\mathcal{E}_R$ is under the assumption of Lemma 1. If $((p^*, s^*), (x_i^*))$ is an Nash-Walras equilibrium of $\mathcal{E}_R$, then (x_i^*) is a Nash-Pareto optimum for the economy $\mathcal{E}_R$.

Proof. Suppose that (x_i^*) is not a Nash-Pareto optimum and let $(x_i') \in \mathcal{F}_R$ be such that for each $i \in I$, $u_i(x_{-i}^*, x_i') \geq u_i(x^*)$ and for some $i_0 \in I$, $u_{i_0}(x_{-i_0}^*, x_{i_0}') > u_{i_0}(x^*)$. From this last condition, given that (x_i^*) is an equilibrium allocation, we know that

$$p^* \cdot x_{i_0}' > p^* \cdot \omega_{i_0} + s^* \cdot \left[r_{i_0} - f(x_{i_0}')\right]. \tag{11}$$

On the other hand, from Lemma 1, we have that for each $i \in I$

$$p^* \cdot x_i' \geq p^* \cdot \omega_i + s^* \cdot [r_i - f(x_i')]. \tag{12}$$

Then, summing and reordering terms, we would obtain that

$$p^* \cdot \sum_{i \in I} x'_i > p^* \cdot \omega + s^* \cdot \left[R - \sum_{i \in I} f(x'_i) \right] > 0 \Leftrightarrow p^* \cdot \left[\omega - \sum_{i \in I} x'_i \right] + s^* \cdot \left[R - \sum_{i \in I} f(x'_i) \right] < 0,$$

which is a contradiction with the feasibility of (x'_i), since all vectors participating in the right side are greater or equal than 0_ℓ and 0_k respectively. This is a contradiction with the equilibrium definition. E.O.P

6 Conclusions

This paper deals with the problem of setting a price system for licences or consumption rights in an economy in which there are caps on consumption, and in order to consume, agents are required to have the corresponding licences for consumption. This fact leads to the possibility of establishing a market of rights or licences.

Examples of this situation are the European Unions Emissions Trading System established in 2005 to reduce greenhouse gas (GHG) concentrations under the Kyoto Protocol (see Ellerman et al. [5]). Also, there are other cap-and-trade systems for emissions that have been implemented in the U.S. In these kinds of systems the price of allowances are set depending on the controlling cost of the negative effects (pollution).

Our model can be used not only on emission control systems, but also to deal with any other licence-based models where allowances are required in advance. Such rights are for instances, licences for aircraft landing at each destination, or for fishing in a region where the amount of captures is regulated. Is it also possible to consider such model to control road congestion by distributing total transit rights for specific links such that flows capacity ratios are limited on these links.

In our approach, agents evaluate their utility considering all the consequences involved in their consumption plan and the consumption plans of the others consumers. The allowances must be acquired at the same time as contracts for raw materials are signed. Thus, prices of the allowances are linked to prices of commodities. Our model is based on the existence of an exogenous function which evaluates the potential negative effects derived from each contract. This mapping associates to every consumption plan (contract) a theoretical amount of rights of each type and consequently, to measure the actual negative effects of consumption is not required in our model. Our aim is to analyze the immediate consequences of setting a cap with a trade system of allowances in a simple model of general equilibrium.

We have shown that under very weak conditions on the fundamentals of the economy, equilibrium exists and, given the *"status quo"*, the equilibrium allocation is Pareto optimum.

Our analysis point out that if the cap is effective for a raw material, the price of this commodity becomes irrelevant at the equilibrium and is the price of the corresponding licence that matters. Given that we deduce that the effectiveness of the cap only depends on the total amount of allowances, the political welfare aspects derived from the distribution of allowances among the agents becomes the relevant problem for the planner of the cap and trade system.

Finally, we would like to remark that in this paper we are not considering the political welfare aspects derived from the distribution of the allowances among the agents. We shall focus on this problem in a future study.

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