ON EXITING AFTER VOTING*

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Abstract

We consider the problem of a society whose members must choose from a finite set of alternatives. After knowing the chosen alternative, members may reconsider their membership in the society by either staying or exiting. In turn, and as a consequence of the exit of some of its members, other members might now find undesirable to belong to the society as well. We analyze the voting behavior of members who take into account the effect of their votes not only on the chosen alternative, but also on the final composition of the society.

Resumen

En este artículo se considera el problema de una sociedad cuyos miembros deben escoger de entre un conjunto finito de alternativas. Después de conocer la alternativa escogida, los miembros pueden reconsiderar su permanencia en la sociedad, decidiendo quedarse o retirarse en la misma. Alternadamente, y como consecuencia de la salida de algunos miembros, otros miembros de la sociedad podrían encontrar ahora no deseable pertenecer a la misma. Se analiza el comportamiento de voto de los miembros, quienes toman en cuenta el efecto

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de sus votos no sólo sobre la alternativa escogida, sino también sobre la composición final de la sociedad.

1. INTRODUCTION

Societies choose alternatives by well-defined voting rules. For instance, political parties and trade unions take up public positions on different issues; communities decide on the contribution level of their members needed to finance common needs; permanent faculty members select new faculty members; scientific societies, and in general democratic societies, choose their representatives. A vast literature on social choice theory studies the properties (in terms of efficiency and incentives, for instance) of alternative voting procedures used to make these choices. Voting by committees, scoring rules, and generalized median voter schemes are examples of specific voting rules used in different settings like those just mentioned (depending on where they operate, different rules might yield different properties of the procedure).

But societies evolve over time. Often, this evolution is triggered off precisely by the chosen alternative: some members might want to leave the society if they feel that the chosen alternative makes their membership undesirable. In turn, other members (although liking the alternative, and even after voting for it) might now find the society undesirable after some of its members have already abandoned it, and so on.

In this paper we want to study explicitly how the possibility that members may exit the society after choosing an alternative affects their voting behavior. We do this by enlarging the set on which preference orderings of members are defined. We assume that members have preference orderings on the set of final societies, where a final society consists of an alternative and a subset of initial members. We consider final societies to be the outcomes of a two stage dynamic game. First, members choose an alternative $x \in X$ by a given voting procedure. Second, and after knowing the chosen alternative, members of the initial society decide whether to stay or exit the society.

While voting procedures are almost always completely described by means of a voting rule, exit procedures are, in contrast, much less regulated by societies. Therefore, we first model exit procedures by a generic family of games, $\{\Gamma(x)\}_{x \in X}$, parametrized by the chosen alternative. Here we want to focus only on voluntary membership in the double sense that members can not be obliged to remain in the society if they do not want to, and members can not be expelled from the society if they want to remain in it. Therefore, we require that each member always has available two strategies, one guaranteeing that he stays in the society and the other guaranteeing that he exits it. In this general setting we exhibit an example in which the dynamic game has no subgame perfect Nash equilibrium in pure strategies. Furthermore, and as particular instances of exit procedures, we consider those in which members decide whether to stay or to exit simultaneously and those in which members take this decision sequentially

and publicly. We show that: (a) If the exit procedure is simultaneous then (a.1) dynamic games, whose voting procedures of the first stage are a large subclass of voting by quota, have Nash equilibria in pure strategies; and (a.2) dynamic games might have multiple Nash equilibria, where in some of them the exit decisions of members exhibit a bad coordination feature: we call these panic equilibria. (b) If the exit procedure is sequential (according to a pre-specified order σ) then (b.1) dynamic games, whose voting procedures of the first stage are all voting by quota, have subgame perfect Nash equilibria in pure strategies; and (b.2), for each $x \in X$, there exists a unique subgame perfect Nash equilibrium in pure strategies of the subgame $\Gamma^\sigma(x)$; but, for a given x, different orderings might generate different equilibria.

Thus, our model is sufficiently general to yield all game theoretic types of difficulties. This richness comes from the fact that a member, when evaluating the consequences of a vote for a particular alternative x, has to take into account (not only whether or not he likes x but also) two simultaneous effects (and their ramifications) of x being chosen. First, the choice of x might be used by member i to get rid of member j if i does not like j and j does not like alternative x (similar and even more involved consequences of x being chosen may arise as well; for instance, i might like j but not j' who belongs to the society just because j is a member of it, but j' would leave it as soon as j exits it; *i.e.*, i votes for x to get rid of j' by bringing about the exit of j). Second, support of alternative x might be used by member i to keep member j'', who is ready to leave the society whenever alternative y is chosen (the chosen one if i does not vote for x), because j'''s membership is critical for i's continued presence in the society (and further obvious effects).

To avoid these difficulties, we proceed by considering only societies whose members have monotonic preference orderings in the sense that each member perceives the membership of all other members as being desirable; namely, for each chosen alternative x, the larger the society, the better it is. Therefore, by assuming monotonic preference orderings we eliminate the possibility that a member votes for an alternative just to get rid of other members. Under this domain restriction we are able to identify, for each chosen alternative x, a very reasonable final society consisting of x and the complementary set of what we call the *exit set after x is chosen*. This set is defined recursively as follows. At each step, all members who would like to leave the society exit it, given that x has been chosen and the current society is formed by all members who in all previous steps wanted to stay. We show that this exit set after x is chosen has many desirable properties. In particular: (a) for any exit procedure it coincides with the set of members that leave the society in any non-panic Nash equilibria and its outcome Pareto dominates the outcomes of the rest of equilibria, (b) for the simultaneous exit procedure it is intimately related to the outcome of the process of iterated elimination of dominated strategies, and (c) for the sequential exit procedure it coincides with the set of members that leave the society in the unique subgame perfect Nash equilibrium of the exit stage.

We finish the paper with one application to the choice problem of a society deciding upon its new members from a given set of candidates. Barberà, Sonnenschein, and Zhou (1991) characterized voting by committees as the class of strategy-proof and onto social choice functions whenever preference orderings of voters are separable or additively representable. In Berga, Bergantiños, Massó,

and Neme (2003) we have already shown that the unique voting by committee that is still strategy-proof, stable,[1] and satisfies voters' sovereignty on the set of candidates is the unanimous rule. Here, we show that any dynamic game in which the voting procedure is a voting by committees without dummies has the feature that voting for a common bad candidate as well as not voting for a common good candidate are dominated strategies, whenever preferences are monotonic and separable in the sense of Barberà, Sonnenschein, and Zhou (1991). Unfortunately, Example 7 shows that the set of undominated strategies of this game might be empty.

Before finishing the Introduction we want to comment on four recent related papers that we have not yet mentioned. The first one is Barberà, Maschler, and Shalev (2001). They study a society that, during a number of periods, might admit in each period a subset of new members. Therefore, at earlier stages voters might not vote only according to whether or not they like a candidate but also according to his tastes (and potential vote) concerning future candidates. For the particular case in which voters partition the set of candidates into two sets, the set of enemies and the set of friends, and the voting rule is voting by quota one (to be admitted it suffices to receive one vote), they identify and study subgame perfect and tremblinghand perfect equilibria with complex dynamic strategic voting behavior. Our paper is different form theirs in many respects but the most important one is that their voters are not able to leave the society, even if all new members are enemies.

Granot, Maschler, and Shalev (2002) study a similar model with expulsion; that is, current members of the society have to decide each period whether to admit new members into the society *and* whether to expel current members of the society for good. They study equilibria for different protocols which depend on whether the expulsion decision has to be taken in each period either simultaneously with, before, or after the admission decision. In contrast, our focus here is on *voluntary exit,* because we find it to be more relevant than expulsion.

Cantala (2002) extends Moulin (1980) by assuming that members have the possibility of excluding themselves from the consumption of the public good if they do not like its chosen level. He shows that the two extreme median voters are the only ones that remain strategy-proof whenever members have single-peaked preference orderings *only* on their corresponding intervals of acceptable levels of public good and they do *not* care about the final set of members consuming the public good. Jackson and Nicolò (2002) departs from Cantala (2002) by letting members care about the number of initial members consuming the public good. They show that strategy-proof and *efficient* social choice functions satisfying an outsider independence condition must be rigid in that they must always choose a fixed number of consumers, regardless of individual desires about consuming the public good.

The paper is organized as follows. We introduce preliminary notation and basic properties of preference orderings in Section 2. Section 3 contains the

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[1]   Stability requires that for any preference profile the social choice function has the property that all members belonging to the final society want to stay (internal stability) and all members who do not belong to the final society do not want to belong (external stability).

description of the voting and exit game, the definition of voting by committees as an example of a voting procedure and the description of general exit procedures. In subsections 3.2.1 and 3.2.2 we describe the simultaneous and sequential exit procedures, respectively, and we present some general results concerning the existence and multiplicity of equilibria. Section 4 is devoted to the case where members have monotonic preference orderings and it contains one application.

2. PRELIMINARIES

Let $N = \{1, ..., n\}$ be the initial set of members of a society that must choose an alternative from a non-empty set X. We assume that n is finite and $n \geq 2$. Generic subsets of N are denoted by S and T, elements of N by i and j, and elements of X by x and y. A *final society* $[S, x]$ consists of the subset of members $S \in 2^N$ that remain in the society, and the chosen alternative $x \in X$. Members have *preferences* over $2^N \times X$, the set of all possible final societies. The *preference relation* of member $i \in N$ over $2^N \times X$, denoted by R_i, is a complete and transitive binary relation. As usual, let P_i and I_i be the strict and indifference preference relations induced by R_i, respectively. We suppose that these preference relations satisfy the following conditions:

(C1) STRICTNESS: For all $x, y \in X$ and $S, T \in 2^N$ such that $i \in S \cap T$ and $[S, x] \neq [T, y]$, either $[S, x] P_i [T, y]$ or $[T, y] P_i [S, x]$.

(C2) INDIFFERENCE: For all $x \in X$ and all $S \in 2^N$, $i \notin S$ if and only if $[S, x] I_i [\varnothing, x]$. Moreover, for all $x, y \in X$, $[\varnothing, x] I_i [\varnothing, y]$.

(C3) NON-INITIAL EXIT: If $\varnothing \in X$, then $[N, \varnothing] P_i [N \setminus \{i\}, \varnothing]$.

STRICTNESS means that member i's preference relation over final societies containing himself is strict. INDIFFERENCE says that member i is indifferent between not belonging to the society and the situation where the society has no members (independently of the chosen alternative). Finally, the NON-INITIAL EXIT condition says that whenever not choosing an alternative is available to the initial society, no member wants to exit.

We denote by $\mathcal{R}_i$ the set of all such preference relations for member i, and by $\mathcal{R}$ the Cartesian product $\mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. Notice that conditions (C1), (C2), and (C3) are member specific and therefore $\mathcal{R}_i \neq \mathcal{R}_j$ for different members i and j. A *preference profile* $R = (R_1, ..., R_n) \in \mathcal{R}$ is a n-tuple of preference relations which we represent sometimes by (R_i, R_{-i}) to emphasize the role of member i's preference relation.

3. THE VOTING AND EXIT GAME

We want to study the equilibrium behavior of members who, when voting, take also into account the effect of the chosen alternative (and hence, the effect

of their votes) on the future composition of the society. To do so, we model this situation as a two stage game

$$\Upsilon = ((M,v) \; ; \; \{\Gamma(x)\}_{x \in X}).$$

Let M_i be the set of possible messages of member i and let $M = M_1 \times \cdots \times M_n$. A voting procedure (M,v) is a mapping $v : M \to X$, where, given the message profile $m = (m_1, ..., m_n) \in M$, the selected alternative is $v(m) \in X$. In the first stage the initial society, by a pre-specified procedure (M,v), has to choose an alternative $x \in X$.

Two examples of voting procedures that use different information (to be provided by members) are the following. Let Q be the set of linear orders on X. A *social choice function* is a voting procedure $ch : Q^N \to X$ in which messages are linear orders on X; that is, each member $i \in N$ declares $Q_i \in Q$ and the alternative $ch(Q_1, ..., Q_n) \in X$ is chosen. A *voting rule* $r : X^N \to X$ is a voting procedure in which each member is required to report an alternative (usually interpreted as his best or "top" alternative); that is, each member $i \in N$ declares $x_i \in X$ and the alternative $r(x_1, ..., x_n) \in X$ is chosen.

After the society has chosen the alternative $x \in X$, using the voting procedure (M,v), each member $i \in N$ reconsiders his membership by taking into account the chosen alternative as well as his expectations on whether or not other members will leave the society. The second stage $\{\Gamma(x)\}_{x \in X}$ corresponds to the exit procedure, which describes what would happen if $x \in X$ were the alternative chosen in the voting stage and the extensive form game $\Gamma(x)$ is played among the set N of members. In contrast with the voting procedure, societies usually do not fully specify neither the rules on how members can leave the society nor the information that should be provided to members about the exit decision of other members. For this reason we do not propose a specific extensive form game, we just require that the strategy sets of each subgame $\Gamma(x)$ have some structure to make sure that exit is voluntary. In Section 4, and under the restriction of monotonic preference relations, we obtain a prediction of the exit procedure which is valid for many classes of extensive form games $\{\Gamma(x)\}_{x \in X}$. The outcome of each subgame $\Gamma(x)$ is a final society; namely, each terminal node of $\Gamma(x)$ is a pair $[S, x]$ where $S \subset N$ represents the set of members who stay in the society. We denote by $E = N \setminus S$ the subset of members leaving the society.

Before proceeding, four general comments are in order.

First, we allow the exit procedure to depend on the alternative chosen in the first stage. This means that it is possible, for instance, that if x is chosen, then members decide simultaneously (and independently) whether they want to stay or to exit the society, while if x' is chosen, then members decide sequentially and publicly, following some pre-specified order, whether to stay or to exit. We believe that the exit procedure should be modelled as being independent of x.[2]

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[2]   Nevertheless, we do not need this assumption to obtain some results in the general framework. Later, we concentrate on two particular cases of exit procedures which are assumed to be independent of x: "simultaneous exit" and "sequential exit". For these two cases we obtain additional results.

Second, we are implicitly assuming that strategies are stationary in the sense that, while they do depend on the alternative chosen in the first stage, they are independent on the ballots (or on some partial information contained on them) with which this alternative is actually chosen. This means that at the beginning of the second stage there are #X subgames, and $\Gamma(x)$ is indeed a subgame for each $x \in X$. Under this assumption a strategy of member $i \in N$ in the game $\Upsilon = ((M, v), \{\Gamma(x)\}_{x \in X})$ can be represented as $b_i = (m_i, \{b_i(x)\}_{x \in X})$ where m_i is the message sent by member i in the voting stage and, for all $x \in X$, $b_i(x)$ is the behavioral strategy played by i in the extensive form $\Gamma(x)$.

Third, in order to maintain the ordinal nature of the preference relations we consider only pure strategies. Let $B_i(x)$ be the set of all pure behavioral strategies of member i in the subgame $\Gamma(x)$ and let B_i be the set of all pure behavioral strategies of member i in Υ. Then, $B_i = M_i \times \{B_i(x)\}_{x \in X}$. As usual, we take $B(x) = B_1(x) \times \cdots \times B_n(x)$ and $B = B_1 \times \cdots \times B_n$. Given $x \in X$ and $b(x) = (b_i(x))_{i \in N} \in B(x)$, $[S(b(x)), x]$ is the final society corresponding to the terminal node of Υ achieved when x was chosen in Stage 1 and members play $b(x)$ in the subgame $\Gamma(x)$.

Fourth, to model voluntary exit, the family of extensive form games $\{\Gamma(x)\}_{x \in X}$ must have the following two properties. First, members can not be forced to stay in the society if they do not want to belong to, and second, members can not be expelled from the society whenever they want to stay. Therefore, we assume that each extensive form game $\Gamma(x)$ has the property that for all $i \in N$ there exist two strategies, $b_i^s(x) \in B_i(x)$ and $b_i^e(x) \in B_i(x)$, such that for all $(b_j(x))_{j \in N \setminus \{i\}} \in (B_j(x))_{j \in N \setminus \{i\}}$,

$$i \in S\left(b_i^s(x), \left(b_j(x)\right)_{j \in N \setminus \{i\}}\right) \text{ and}$$

$$i \notin S\left(b_i^e(x), \left(b_j(x)\right)_{j \in N \setminus \{i\}}\right).$$

We are interested in the subgame perfect Nash equilibria of Υ. Example 1 below shows that the game Υ might not have Subgame Perfect Nash Equilibria in pure strategies (*SPNE*).

Example 1 Let $N = \{1, 2\}$ be a society choosing one alternative from the set $X = \{y, z\}$. Define the voting procedure (M, v) by letting $M_1 = M_2 = X$, $v(y, y) = v(z, z) = y$, and $v(z, y) = v(y, z) = z$. No further restrictions are made on $\Gamma(y)$ and $\Gamma(z)$. The preference relation of member 1 is

$$[N, y]\, P_1\, [\{1\}, y]\, P_1\, [N, z]\, P_1\, [\{1\}, z]\, P_1\, [\varnothing, y],$$

and, by condition (C2), the rest of pairs $[T, x]$ with $T \subset N$ and $x \in X$ satisfy $[T, x]$ $I_1\, [\varnothing, y]$.

The preference relation of member 2 is

$$[N, z]\, P_2\, [\{2\}, z]\, P_2\, [N, y]\, P_2\, [\{2\}, y]\, P_2\, [\varnothing, y],$$

and again, by condition (C2), the rest of pairs $[T, x]$ with $T \subset N$ and $x \in X$ satisfy $[T, x] \, I_2 \, [\varnothing, y]$.

Consider first the subgame $\Gamma(y)$. There are four possible terminal nodes of Υ when $\Gamma(y)$ is reached: $[\varnothing, y]$, $[\{1\}, y]$, $[\{2\}, y]$, and $[N, y]$. Since

$$[N, y] \, P_1 \, [\{1\}, y] \, P_1 \, [\varnothing, y] \, I_1 \, [\{2\}, y],$$

the existence of the behavioral strategy $b_i^s(y)$ guarantees that in any *SPNE* of Υ, after y is chosen, only the terminal nodes $[N, y]$ and $[\{1\}, y]$ can be reached. Using a symmetric argument for member 2 we conclude that in any *SPNE* of Υ, after y is chosen, only the terminal nodes $[N, y]$ and $[\{2\}, y]$ can be reached. Then, in any *SPNE* of Υ, after y is chosen, the final society is $[N, y]$. Similarly, we conclude that in any *SPNE* of Υ, after z is chosen, the final society is $[N, z]$. But the following normal form game (corresponding to the voting stage, after taking into account that no exit will be induced by any *SPNE* strategy in any of the two subgames)

1\2	y	z
y	$[N, y]$	$[N, z]$
z	$[N, z]$	$[N, y]$

has no Nash Equilibrium in pure strategies (*NE*). Hence, Υ has no *SPNE*.

Remark 1 This example shows that the non existence of *SPNE* of Υ is a consequence of the non existence of *NE* in the voting procedure of the first stage.

We now introduce a family of voting procedures that we will use intensively in the sequel: voting by committees.

3.1. Choosing new members: voting by committees

Consider a society choosing, from a given set K of candidates, a subset of its new members. In this case, the set of social alternatives X is the family of all subsets of the set of candidates K; that is, $X = 2^K$. Voting by committees are defined by a collection of families of winning coalitions (committees), one for each candidate. Founders vote for sets of candidates. To be elected, a candidate must get the vote of all members of some coalition among those that are winning for that candidate. Formally, a *committee* $\mathcal{W}$ is a non-empty family of non-empty coalitions of N satisfying coalition monotonicity ($S \in \mathcal{W}$ and $S \subset T$ implies $T \in \mathcal{W}$). Given a committee $\mathcal{W}$ we denote the set of minimal winning coalitions by $\mathcal{W}^m \equiv \{S \in \mathcal{W} \mid T \notin \mathcal{W} \text{ for all } T \subsetneq S\}$.

Following Barberà, Sonnenschein, and Zhou (1991) we say that a social choice function $ch : Q^N \rightarrow 2^K$ is *voting by committees* if for each $k \in K$, there exists a committee $\mathcal{W}_k$ such that for all $(Q_1, ..., Q_n) \in Q^N$,

$$k \in ch(Q_1, ..., Q_n) \text{ if and only if } \{i \in N \mid k \in t(Q_i)\} \in \mathcal{W}_k,$$

where, for each $i \in N$, $t(Q_i)$ denotes the best alternative according to the linear order Q_i. Observe that voting by committees have the tops-only property since they only depend on the vector of best subsets. Accordingly, and with an slight abuse of language, we will directly treat voting by committees as voting rules.

Let $vc : (2^K)^N \to 2^K$ be voting by committees and let $\mathcal{W}^m = (\mathcal{W}_k^m)_{k \in K}$ be its corresponding families of minimal winning coalitions. We say that vc has no *dummies* if for all $k \in K$ and all $i \in N$ there exists $S \in \mathcal{W}_k^m$ such that $i \in S$.

We now present the special subclass of anonymous and neutral voting by committees, corresponding to those committees whose set of winning coalitions of all candidates are equal and they depend only on their cardinality. For each set S denote by $\#S$ the number of elements of S. Given an integer $1 \le q \le n$, we say that the voting rule $vc^q : (2^K)^N \to 2^K$ is *voting by quota* q if for all $(x_1, ..., x_n) \in (2^K)^N$ and $k \in K$,

$$k \in vc^q(x_1, ..., x_n) \text{ if and only if } \#\{i \in N \mid k \in x_i\} \ge q.$$

Barberà, Sonnenschein, and Zhou (1991) characterize the family of voting by committees as the class of all strategy-proof and onto social choice functions on the domain of additive (as well as separable) preferences. In addition, they characterize voting by quota as the class of all strategy-proof, anonymous, neutral, and onto social choice functions on these two domains of preferences.

Next we present two alternative and polar cases of exit procedures.

3.2. Exit procedures

The *simultaneous exit* procedure, modelling situations when exit is a private decision which is kept private (for instance, when the membership has to be renewed yearly by just sending a check to the secretary of the society), and the *sequential exit* procedure, modelling situations where membership is public (for instance, when leader A of a political party announces publicly that he is leaving the party due to disagreements with the official position taken by the party on a particular issue; this in turn might also produce further public announcements of other leaders leaving the party due to the exit of leader A *and/or* to disagreements with the official position, and so on).

3.2.1. Simultaneous exit

We now consider societies where, after knowing that alternative $x \in X$ has been chosen by the voting procedure (M, v), each member of the society reconsiders, independently and simultaneously, his membership. Then, for all $x \in X$, $\Gamma(x)$ is the extensive form game in which members select, independently and simultaneously, an element of $\{e, s\}$. Therefore, $B_i(x) = \{e, s\}$ for all $i \in N$ and $x \in X$. Moreover, given $b(x) \in B(x)$, $S(b(x)) = \{i \in N \mid b_i(x) = s\}$ and $E(b(x)) = \{i \in N \mid b_i(x) = e\}$.

The following example shows that, even if the voting procedure for choosing new members is voting by quota 1, the set of *SPNE* of Υ with simultaneous exit might be empty.

Example 2 Let $N = \{1, 2, 3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member of the society (*i.e.*, $X = \{\varnothing, y\}$). Assume that the voting procedure ($\{\varnothing, y\}^N$, vc^1) is voting by quota 1 and the exit procedure is simultaneous. Consider the preference profile $R = (R_1, R_2, R_3)$ $\in \mathcal{R}$, additively representable by the following table

	u_1	u_2	u_3
1	1	−2	1
2	2	16	2
3	3	−10	3
y	−5	−5	100

where the number in each cell represents the utility each member $i \in N$ assigns to members in N, as well as to candidate y (we normalize by setting $u_i(\varnothing) = 0$ for all $i \in N$). That is, for all $i \in N$, all $x, x' \in \{\varnothing, y\}$, and all $T, T' \in 2^N$, $[T, x] P_i$ $[T', x']$ if and only if

$$\begin{cases} \sum_{j \in T} u_i(j) + u_i(x) > \sum_{j \in T'} u_i(j) + u_i(x') & \text{if } i \in T \cap T' \\ \sum_{j \in T} u_i(j) + u_i(x) > 0 & \text{if } i \in T \text{ but } i \notin T'. \end{cases}$$

Notice that, by the indifference condition (C2), if $i \notin T$ and $i \notin T'$ then, $[T, x]$ $I_i [T', x']$. Again, we normalize by saying that if $i \notin T$ then, the utility of $[T, x]$ is 0.

First, observe that s is an strictly dominant action for member 3 in $\Gamma(\varnothing)$ and $\Gamma(y)$. Thus, in any *SPNE* strategy b of Υ, $b_3(\varnothing) = b_3(y) = s$. Assume that $b = (m, \{b(x)\}_{x \in X}$) is a *SPNE* of Υ but $y \neq vc^1(m)$. Then $vc^1(m) = \varnothing$. Consider the strategy $b'_3 = (m'_3, b'_3(\varnothing), b'_3(y))$ of member 3 where, $m'_3 = y$ and $b'_3(\varnothing) = b'_3(y) = s$. Then, and since the voting procedure is voting by quota 1, $vc^1(m'_3, m_{-3}) = y$ and $3 \in S(b'_3(y), b_{-3}(y))$. But $[S(b'_3(y), b_{-3}(y)), y] P_3 [S(b(\varnothing)), \varnothing]$ contradicts that b is a *SPNE* of Υ. Therefore, in any *SPNE* of Υ candidate y is admitted and member 3 stays in the society. Now the simultaneous strategic decisions of members 1 and 2 in the subgame $\Gamma(y)$ can be represented by the following normal form game

1\2	e	s
e	0, 0	0, 1
s	−1, 0	1, −1

which does not have *NE*. Hence, Υ does not have *SPNE*.

Remark 2 The situation faced by members 1 and 2 in $\Gamma(y)$ can be easily translated to any voting procedure (M, v). Example 2 shows that this may even happen on the equilibrium path.

Nevertheless, it is possible to find interesting subclasses of voting procedures for which the game Υ has *NE*, when exit is simultaneous. Proposition 1 below identifies some of them.

Proposition 1 *Assume that the voting procedure for choosing new members is voting by quota q, with $q \geq 2$, and $\Gamma(x)$ is simultaneous exit for all $x \in 2^K$. Then,*

the game $\Upsilon = \left(\left(\left(2^K \right)^N, vc^q \right), \{\Gamma(x)\}_{x \in 2^K} \right)$ *has NE.*

Proof Let $b = (b_1, ..., b_n) \in B$ be such that for all $i \in N$, $b_i = (m_i, \{b_i(x)\}_{x \in X})$ is such that $m_i = \emptyset$ and $b_i(x) = s$ for all $x \in 2^K$. Then, $[S(b(vc^q(m))), vc^q(m)] = [N, \emptyset]$. To prove that b is a *NE* of Υ, consider any $i \in N$ and let $b'_i = (m'_i, \{b'_i(x)\}_{x \in X}) \in B_i$ be arbitrary. Since the voting procedure is voting by quota $q \geq 2$, and $m_j = \emptyset$ for all $j \in N \setminus \{i\}$, $vc^q(m'_i, m_{-i}) = \emptyset$. If $b'_i(\emptyset) = s$ then

$$[S(b'_i(vc^q(m'_i, m_{-i})), b_{-i}(vc^q(m'_i, m_{-i}))), vc^q(m'_i, m_{-i})] = [N, \emptyset] = [S(b(vc^q(m))), vc^q(m)],$$

which means that member i does not improve by playing b'_i. If $b'_i(\emptyset) = e$ then $[S(b'_i(vc^q(m'_i, m_{-i})), b_{-i}(vc^q(m'_i, m_{-i}))), vc^q(m'_i, m_{-i})] = [N \setminus \{i\}, \emptyset]$. By the noninitial exit condition (C3), $[N, \emptyset] \, P_i \, [N \setminus \{i\}, \emptyset]$, which means that member i does not improve either. Hence, b is a *NE* of Υ. ∎

Besides its potential non-existence problem, the game Υ with simultaneous exit might have *SPNE* in which members exit the society only because they think that other members will exit as well, but all of them would prefer that all stay. We call them panic equilibria. Formally, we say that $b(x)$ is a *panic* equilibrium of $\Gamma(x)$ if $b(x)$ is a *SPNE* of $\Gamma(x)$ and there exists another *SPNE* strategy $b'(x)$ such that $S(b'(x)) \supsetneq S(b(x))$. Since $b'(x)$ is a *SPNE* observe that members in $T = S(b'(x)) \setminus S(b(x)) \neq \emptyset$ prefer to stay with members of $S(b(x))$ ($[T \cup S(b(x)), x] \, P_i \, [\emptyset, x]$ for all $i \in T$); members in $S(b(x))$ do not want to exit when members of T stay ($[T \cup S(b(x)), x] \, P_j \, [\emptyset, x]$ for all $j \in S(b(x))$); and members in $E(b'(x))$ exit when members in $S(b'(x))$ stay ($[\emptyset, x] \, P_{i'} \, [S(b'(x)) \cup \{i\}, x]$ for all $i' \in E(b'(x))$). We know that $b_i(x) = b'_i(x) = s$ for all $i \in S(b(x))$, $b_i(x) = b'_i(x) = e$ for all $i \in E(b'(x))$, and $b_i(x) = e$ and $b'_i(x) = s$ for all $i \in T$. Assume that $T = \{i\}$. Since $b(x)$ is a *SPNE*, $[S(b(x)), x] \, R_i \, [S(s, b_{-i}(x)), x] = [S(b(x)) \cup \{i\}, x]$. By condition (C2), $[S(b(x)), x] \, P_i \, [S(b(x)) \cup \{i\}, x]$. But this is a contradiction because $b'(x)$ is a *SPNE* and $b'(x) = (s, b_{-i}(x))$. Then, $\#T \geq 2$ and hence we can make the "panic" interpretation of the equilibrium $b(x)$: each member $i \in T$ exits the society because he thinks that members in $T \setminus \{i\}$ will do so. Nevertheless, all members in T prefer that all of them remain in the society.

The next example shows that panic equilibria can indeed exist.

Example 3 Consider again Example 2 except that now the preference profile *R* is representable by the following table

	u_1	u_2	u_3
1	5	8	1
2	8	5	2
3	−10	−10	3
y	1	1	100

Let $b = (b_1, b_2, b_3)$ be such that $m = (y, y, y)$ and $b(\emptyset) = b(y) = (e, e, s)$. It is immediate to see that b is a *SPNE* and $[S(b(vc^1(m)))\,,\,vc^1(m)] = [\{3\}, y]$.

We check that b is a panic equilibria. Member 1 (2) leaves the society because given that member 2 (1) leaves it, his best response is also to leave it. But there is no a reasonable support for these beliefs. If candidate *y* is not admitted, the society is in the same situation as it has been before voting and every member prefers to stay rather than to leave. If candidate y is admitted as a new member of the society the situation is even better from the point of view of all members then, why to leave?

Now, let $b' = (b'_1, b'_2, b'_3)$ be such that $m' = (y, y, y)$ and $b'(\emptyset) = b'(y) = (s, s, s)$. Then, b' is also a *SPNE* but it is not a panic equilibrium. In this case $[S(b'(vc^1(m'))), vc^1(m')] = [N, y]$. We believe that this is the reasonable solution of this example. In Section 4 we show that, if preference relations are monotonic, no panic equilibrium exits.

3.2.2. Sequential exit

We consider now societies whose members, after knowing that the alternative $x \in X$ has already been chosen, reconsiders, sequentially and knowing the decision already taken by their predecessors, their membership of the society.

Let $\sigma : \{1, ..., n\} \to N$ be a one-to-one mapping representing this order; namely, $\sigma(t) = i$ means that member i is in the tth position according to the ordering σ. Obviously, there are $n!$ different orderings. Denote by Σ the set of all possible orderings. We denote by $Pre(i, \sigma)$ the set of predecessors of member i in σ. Then,

$$Pre(i, \sigma) = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}.$$

Given $\sigma \in \Sigma$ we consider the exit procedure where for all $x \in X$, $\Gamma^\sigma(x)$ is the extensive form game in which each member, sequentially (in the order given by σ) and knowing the decision of his predecessors, selects an element of $\{e, s\}$. If member i chooses e (exit) he is not in the final society whereas if he chooses s (stay) he is a member of the final society.

Again, the exit procedure is independent of the chosen alternative. In all subgames members decide in the order given by σ. Let $\Upsilon^\sigma = ((M, v), \{\Gamma^\sigma(x)\}_{x \in X})$ be the two stage game associated with the ordering σ.

To describe the set of pure behavioral strategies of members, take $i \in N$, $x \in X$, and $\sigma \in \Sigma$. Because, when member i must take his decision in $\Gamma^\sigma(x)$, he knows the decisions already taken by the members of set $Pre(i, \sigma)$, we can identify the information sets of member i with $2^{Pre(i,\sigma)}$, the family of subsets of $Pre(i, \sigma)$, where any $T \in 2^{Pre(i,\sigma)}$ represents the set of members of $Pre(i, \sigma)$ who have already decided to stay in the society. Thus, we can write the set of pure behavioral strategies of member i in $\Gamma^\sigma(x)$ as

$$B_i^\sigma(x) = \left\{ b_i(x) : 2^{Pre(i,\sigma)} \to \{e, s\} \right\}.$$

The next proposition shows that with sequential exit, and for any alternative x and any ordering σ, the subgame $\Gamma^\sigma(x)$ has always a *SPNE*. Moreover, this equilibrium is unique. This constitutes an obvious advantage of sequential exit over simultaneous exit.

Proposition 2 *For all $x \in X$ and all $\sigma \in \Sigma$ the subgame $\Gamma^\sigma(x)$ has a unique SPNE.*

Proof Take $x \in X$ and assume, without loss of generality, that $\sigma(i) = i$ for all $i \in N$. Let $T \in 2^{Pre(n,\sigma)}$ be an information set of member n. If n exits, $\Gamma^\sigma(x)$ ends in the terminal node $[T, x]$. If n stays, $\Gamma^\sigma(x)$ ends in the terminal node $[T \cup \{n\}, x]$. By the indifference condition (C2), either $[T, x] \, P_n \, [T \cup \{n\}, x]$ or $[T \cup \{n\}, x] \, P_n \, [T, x]$. Therefore, in any *SPNE* of $\Gamma^\sigma(x)$ the strategy of member n is

$$\left(b_n^*(x) \right)(T) = \begin{cases} e & \text{if } [T, x] \, P_n \, [T \cup \{n\}, x] \\ s & \text{if } [T \cup \{n\}, x] \, P_n \, [T, x]. \end{cases}$$

Next, let $T \in 2^{Pre(n-1,\sigma)}$ be an information set of member $n - 1$. If $n - 1$ exits, $\Gamma^\sigma(x)$ ends in the terminal node $[T', x]$ such that $n - 1 \notin T'$. If $n - 1$ stays, $\Gamma^\sigma(x)$ ends in the terminal node $[T'', x]$ where $T'' = T \cup \{n - 1\}$ if $(b_n^*(x)) (T \cup \{n - 1\}) = e$ and $T'' = T \cup \{n - 1, n\}$ if $(b_n^*(x)) (T \cup \{n - 1\}) = s$. By the indifference condition (C2), either $[T', x] \, P_{n-1} \, [T'', x]$ or $[T'', x] \, P_{n-1} \, [T', x]$. Therefore, in any *SPNE* of $\Gamma^\sigma(x)$ the strategy of member $n - 1$ is

$$\left(b_{n-1}^*(x) \right)(T) = \begin{cases} e & \text{if } [T', x] \, P_{n-1} \, [T'', x] \\ s & \text{if } [T'', x] \, P_{n-1} \, [T', x]. \end{cases}$$

Now, and since $\Gamma^\sigma(x)$ has perfect information, using a conventional backwards induction argument together with the indifference condition (C2), the existence of a unique *SPNE* strategy $b^*(x)$ of $\Gamma^\sigma(x)$ follows.

Since for each $x \in X$ the subgame $\Gamma^\sigma(x)$ has a unique *SPNE* strategy $b^*(x)$, it is not possible to find another *SPNE* strategy $b'(x)$ such that $S(b'(x)) \supsetneq S(b(x))$. Therefore, there is no panic equilibria with sequential exit, which is another advantage of the sequential exit over the simultaneous exit.

We obtain now a similar result to Proposition 1 for sequential exit.

Proposition 3 *Assume that the voting procedure for choosing new members is voting by quota q and for all $x \in 2^K$, $\Gamma^\sigma(x)$ is the sequential exit procedure associated to an ordering σ. Then, the game $\Upsilon^\sigma = \left(\left(\left(2^K \right)^N, vc^q \right), \left\{ \Gamma^\sigma(x) \right\}_{x \in 2^K} \right)$ has SPNE.*

Proof Fix $\sigma \in \Sigma$ and assume that $q \geq 2$. Let $b^* = (b_1^*, ..., b_n^*) \in B$ be such that for all $i \in N$, $b_i^* = \left(m_i^*, \left\{ b_i^*(x) \right\}_{x \in 2^K} \right)$ is such that $m_i^* = \emptyset$ and for all $x \in 2^K$, $b_i^*(x)$ is the unique *SPNE* of the subgame $\Gamma^\sigma(x)$ given by Proposition 2. It is straightforward to prove that $[S(b^*(vc^q(m^*))), vc^q(m^*)] = [N, \emptyset]$. Using arguments similar to those already used in the proof of Proposition 1 we can show that b^* is a *SPNE* of Υ^σ.

Assume that $q = 1$. Remember that $X = 2^K$ and $n \geq 2$. Let $b^* = (b_1^*, ..., b_n^*) \in B$ be such that for all $i \in N$, $b_i^* = (m_i^*, \{b_i^*(x)\}_{x \in X})$ is such that $m_i^* = K$ and for all $x \in 2^K$, $b_i^*(x)$ is the unique *SPNE* of the subgame $\Gamma^\sigma(x)$ given by Proposition 2. Then, $vc^1(m^*) = K$. We now prove that b^* is a *SPNE* of Υ^σ. By definition of $\left\{ b_i^*(x) \right\}_{x \in 2^K}$ we know that b^* induces a *SPNE* in any subgame starting at x in the second stage of Υ^σ. Then, it only remains to be proven that b^* is a *NE* of Υ^σ. Take $i \in N$ and $b_i' = (m_i', \{b_i'(x)_{x \in 2^K}) \in B_i$. Since $m_j^* = K$ for all $j \in N \setminus \{i\}$ and $q = 1$ we conclude that $vc^1(m_i', m_{-i}^*) = K$ for all m_i'. Thus, $[S(b^*(vc^1(m^*))), vc^1(m^*)]$ $R_i [S(b_i'(vc^1(m_i', m_{-i}^*)), b_{-i}^*(vc^1(m_i', m_{-i}^*))), vc^1(m_i', m_{-i}^*)]$ because $vc^1(m_i', m_{-i}^*) = vc^1(m^*) = K$ and $b^*(K)$ is a *SPNE* of $\Gamma^\sigma(K)$. This means that member i can not improve by playing b_i' instead of b_i^*.

Observe that the result of Proposition 3 is more general than the one established in Proposition 1 because now q is arbitrary and existence is in terms of *SPNE*, instead of *NE*.

The next example shows that the *SPNE* of $\Gamma^\sigma(x)$ and $\Gamma^{\sigma'}(x)$ (with $\sigma \neq \sigma'$) might differ. Therefore, the outcomes of *SPNE* of Υ^σ and $\Upsilon^{\sigma'}$ corresponding to different orderings σ and σ' might differ too.

Example 4 Consider again Example 2 except that now the exit procedure is sequential exit and the preference profile R is representable by the following table

	u_1	u_2	u_3
1	5	–8	8
2	5	10	–15
3	5	5	10
y	–20	–12	2

Consider the orderings σ and σ', where $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $\sigma'(1) = 1$, $\sigma'(2) = 3$, and $\sigma'(3) = 2$. By Proposition 2 we know that $\Gamma^\sigma(y)$ and $\Gamma^{\sigma'}(y)$ have a unique *SPNE*, which we denote by $b^\sigma(y)$ and $b^{\sigma'}(y)$, respectively. It is easy to see that $S(b^\sigma(y)) = \{3\}$, but $S(b^{\sigma'}(y)) = \emptyset$. Given that the voting procedure is

voting by quota 1 it is straightforward to prove that $\Upsilon^{\sigma'}$ has two *SPNE* outcomes: $[\varnothing, y]$ and $[N, \varnothing]$. The first one can be obtained, for instance, with b such that $b_i = (y, \{b_i^\sigma(x)\}_{x \in \{\varnothing, y\}})$ for all $i \in N$. The second one with $\hat{b}$ such that $\hat{b}_i = (\varnothing, \{b_i^{\sigma'}(x)\}_{x \in \{\varnothing, y\}})$ for all $i \in N$. It seems to us that $[N, \varnothing]$ is the most reasonable solution. Nevertheless, Υ^σ has only one *SPNE* outcome: $[\{3\}, y]$. This corresponds, for instance, to a $\bar{b}$ such that $\bar{b}_3 = (y, \{b_3^\sigma(x)\}_{x \in \{\varnothing, y\}})$ and $\bar{b}_i = (\varnothing, \{b_i^\sigma(x)\}_{x \in \{\varnothing, y\}})$ for all $i \in N \setminus \{3\}$.

Example 4 shows that details of the exit procedure might have important effects on the final outcome. For instance, under σ member 3 has incentives to vote for y but under σ' his incentives are just the opposite. Moreover, this is not specific to the problem of choosing new members. It is easy to adapt Example 4 to other settings, for example, choosing a level of a public good as in Moulin (1980).

A natural question arises: is it possible to find a plausible subdomain of preferences where the outcome of the sequential exit game is independent of the ordering? In the next section we give an affirmative answer to this question.

4. MONOTONIC PREFERENCES

There are many societies whose members consider the exit of other members undesirable, independently of the chosen alternative. For instance, scientific societies want to become larger, political parties do not want to lose affiliates, the United Nations want to have as members as many countries as possible, and so on. We call the preference relations that satisfy this general condition monotonic. Formally,

MONOTONICITY: A preference relation $R_i \in \mathcal{R}_i$ is *monotonic* if for all $x \in X$ and all $T \subsetneq T' \subset N$ such that $i \in T$,

$$[T', x] \, P_i \, [T, x].$$

A preference profile $R = (R_1, ..., R_n) \in \mathcal{R}$ is said to be *monotonic* if the preference relation R_i is monotonic for all $i \in N$.

Notice that monotonicity does not impose any condition when comparing two final societies with different chosen alternatives. In particular, monotonicity admits the possibility that member i prefers to belong to a smaller society; namely, the ordering $[T, x] \, P_i \, [T', x']$ with $i \in T \subsetneq T'$ is compatible with monotonicity, as long as $x \neq x'$. But monotonicity also admits that member i prefers to exit the society if the chosen alternative is perceived as being very bad; namely, $[\varnothing, x] \, P_i \, [N, x]$ is compatible with monotonicity too.

We now define the set $E(x)$ as the subset of members leaving the society after the alternative $x \in X$ has been chosen. We argue that, independently of the exit procedure $\Gamma(x)$, the set $E(x)$ is a good prediction of the exit generated by the choice of x. The definition of $E(x) \subset N$ is recursive and as follows.

First define the set $E^1(x)$ as the set of members who want to leave the society, when x is chosen, even when the rest of the members remain in the society. Formally,

$$E^1(x) = \{i \in N \mid [N \setminus \{i\}, x] \, P_i \, [N, x]\}.$$

Remember that by the indifference condition (C2) if $i \notin T$ then $[T, x] \, I_i \, [\varnothing, x]$. Therefore, $E^1(x)$ can be rewritten as $\{i \in N \mid [\varnothing, x] \, P_i \, [N, x]\}$.

Let $t \geq 1$ and assume $E^{t'}(x)$ hass been defined for all t' such that $1 \leq t' \leq t$. Then,

$$E^{t+1}(x) = \left\{ i \in N \setminus \left(\bigcup_{t'=1}^{t} E^{t'}(x) \right) \mid [\varnothing, x] \, P_i \left[N \setminus \left(\bigcup_{t'=1}^{t} E^{t'}(x) \right), x \right] \right\}.$$

Let t_x be either equal to 1 if $E^1(x) = \varnothing$ or else be the smallest positive integer satisfying the property that $E^{t_x}(x) \neq \varnothing$ but $E^{t_x+1}(x) = \varnothing$. Notice that t_x is well defined and $t_x \leq n$. Then, define the *exit set after x* as

$$E(x) = \bigcup_{t=1}^{t_x} E^t(x).$$

Notice that $E(x)$ depends only on the preference profile R but it is completely independent of the exit procedure used in the second stage of Υ.

The following example illustrates the definition of $E(x)$ and suggests some of its properties established in Proposition 4 below.

Example 5 Let $N = \{1, 2, 3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member of the society (*i.e.*, $X = \{\varnothing, y\}$). Consider first the preference profile $R \in \mathcal{R}$, additively representable (as in Example 2) by the following table:

	u_1	u_2	u_3
1	7	6	1
2	6	2	2
3	15	15	3
y	−10	−10	−15

By monotonicity, $E(\varnothing) = \varnothing$. However, $E^1(y) = \{3\}$, $E^2(y) = \{2\}$, $E^3(y) = \{1\}$, and $E^4(y) = \varnothing$. Thus, $E(y) = \{1, 2, 3\}$.

Let $(R_1, R_2', R_3) \in \mathcal{R}$ be a preference profile, additively representable by the new table:

	u_1	u_2'	u_3
1	7	7	1
2	6	6	2
3	15	15	3
y	−10	−10	−15

Now, $E^1(y) = \{3\}$ and $E^2(y) = \emptyset$. Thus, $E(y) = \{3\}$. The extensive form $\Gamma(y)$ with simultaneous exit has two *SPNE*: $b(y) = (s, s, e)$, inducing the final society $[\{1, 2\}, y]$, and $b'(y) = (e, e, e)$, inducing $[\emptyset, y]$. Observe that the two *SPNE* have the property that the exit they induce contains $E(y)$; one of these *SPNE*, $b(y)$, induces exactly $E(y)$; and $b(y)$ is the only non-panic equilibrium.

The next Proposition states that, independently of the exit procedure defining the extensive form game $\Gamma(x)$, members of $E(x)$ always exit in any *NE* of $\Gamma(x)$. Moreover, it is always possible to find a *NE* strategy $b(x)$ of $\Gamma(x)$ such that the exit set after x coincides with the set $E(b(x))$ of members that exit when $b(x)$ is played.

Proposition 4 *Let $x \in X$ be given and let $\Gamma(x)$ be any exit procedure. Then:*

(a)   *$E(x) \subset E(b(x))$ for all NE strategy $b(x)$ of $\Gamma(x)$.*

(b)   *There exists a NE strategy $b(x)$ of $\Gamma(x)$ such that $E(b(x)) = E(x)$.*

Proof (a) Let $b(x)$ be a *NE* of $\Gamma(x)$. We proceed by induction.

We first prove that $E^1(x) \subset E(b(x))$. Suppose not. Assume $E^1(x) \neq \emptyset$ and there exists $i \in E^1(x)$ such that $i \notin E(b(x))$. Since $S(b(x)) = N \setminus E(b(x))$, $i \in S(b(x))$. Assume member i plays $b_i^e(x)$ instead of $b_i(x)$. Then,

$$[S(b(x)), x] \, R_i \, [S(b_i^e(x), b_{-i}(x)), x]$$

because $b(x)$ is a *NE* of $\Gamma(x)$. By definition of $b_i^e(x)$ we know that $i \notin S(b_i^e(x), b_{-i}(x))$. By the indifference condition (C2),

$$[S(b(x)), x] \, P_i \, [\emptyset, x].$$

Since preferences are monotonic, $[N, x] \, R_i \, [S(b(x)), x]$, and hence, $[N, x] \, P_i \, [\emptyset, x]$. But this is a contradiction because, by definition of $E^1(x)$ and the fact that $i \in E^1(x)$, $[\emptyset, x] \, P_i \, [N, x]$. Then, $i \in E(b(x))$.

Assume that $E^{t'}(x) \subset E(b(x))$ for all $t' \leq t < t_x$. We now prove that $E^{t+1}(x) \subset E(b(x))$. Suppose not. There exists $i \in E^{t+1}(x)$ such that $i \notin E(b(x))$ and hence, $i \in S(b(x))$. Since $b(x)$ is a *NE* of $\Gamma(x)$, if member i plays $b_i^e(x)$ instead of $b_i(x)$,

$$[S(b(x)), x] \, R_i \, [S(b_i^e(x), b_{-i}(x)), x].$$

By definition of $b_i^e(x)$ we know that $i \notin S(b_i^e(x), b_{-i}(x))$. By the indifference condition (C2),

$$[S(b(x)), x] \, P_i \, [\emptyset, x].$$

By the induction hypothesis, $\bigcup_{t'=1}^{t} E^{t'}(x) \subset E(b(x))$. Thus, $S(b(x)) = N \setminus E(b(x))$

$\subset N \setminus \left(\bigcup_{t'=1}^{t} E^{t'}(x) \right)$. Since preferences are monotonic,

$$\left[N \setminus \left(\bigcup_{t'=1}^{t} E^{t'}(x) \right), x \right] R_i \left[S(b(x)), x \right].$$

Therefore,

$$\left[N \setminus \left(\bigcup_{t'=1}^{t} E^{t'}(x) \right), x \right] P_i \left[\varnothing, x \right]$$

But this contradicts the fact that $i \in E^{t+1}(x)$. Therefore, $E^{\bar{t}}(x) \subset E(b(x))$ for

all $\bar{t} \le t_x$. Hence, $E(x) \subset E(b(x))$ because $E(x) = \bigcup_{\bar{t}=1}^{t_x} E^{\bar{t}}(x)$.

(b) Let $b(x) \in B(x)$ be such that $b_i(x) = b_i^e(x)$ for all $i \in E(x)$ and $b_i(x) = b_i^s(x)$ for all $i \in N \setminus E(x)$. We prove that $b(x)$ is a NE of $\Gamma(x)$ such that $E(b(x)) = E(x)$. By definition of $b_i^e(x)$ and $b_i^s(x)$, $E(b(x)) = E(x)$ and $S(b(x)) = N \setminus E(x)$. We now prove that $b(x)$ is a NE of $\Gamma(x)$.

First take $i \in E(x)$ and let $b_i'(x) \in B_i(x)$ be arbitrary. We know that $N \setminus E(x) \subseteq S(b_i'(x), b_{-i}(x))$ and $E(x) \setminus \{i\} \subseteq E(b_i'(x), b_{-i}(x))$. If $i \in E(b_i'(x), b_{-i}(x))$ then, $[S(b_i'(x), b_{-i}(x)), x] = [S(b(x)), x]$, which means that member i cannot improve. Assume that $i \notin E(b_i'(x), b_{-i}(x))$. Then, $S(b_i'(x), b_{-i}(x)) = (N \setminus E(x)) \cup \{i\}$. Since $i \in E(x)$ there exists t, $1 \le t \le t_x$, such that $i \in E^t(x)$. Hence,

$$[\varnothing, x] \, P_i \left[N \setminus \left(\bigcup_{t'=1}^{t-1} E^{t'}(x) \right), x \right]$$

Since preferences are monotonic,

$$\left[N \setminus \left(\bigcup_{t'=1}^{t-1} E^{t'}(x) \right), x \right] P_i \left[(N \setminus E(x)) \cup \{i\}, x \right].$$

Thus, member i cannot improve either, because, by the indifference condition (C2), $[\varnothing, x] \, I_i \, [S(b(x)), x]$. Now take $i \notin E(x)$ and let $b_i'(x) \in B_i(x)$ be arbitrary. We know that $N \setminus (E(x) \cup \{i\}) \subseteq S(b_i'(x), b_{-i}(x))$ and $E(x) \subseteq E(b_i'(x), b_{-i}(x))$. If $i \notin E(b_i'(x), b_{-i}(x))$ then $[S(b_i'(x), b_{-i}(x)), x] = [S(b(x)), x]$, which means that member i cannot improve. Assume that $i \in E(b_i'(x), b_{-i}(x))$. Then, $[S(b_i'(x), b_{-i}(x)), x] \, I_i \, [\varnothing, x]$ by the indifference condition (C2). Since $i \notin E(x)$ and $i \notin E^{t_x+1}(x)$ we conclude $[N \setminus E(x), x] \, P_i \, [\varnothing, x]$. Thus, member i cannot improve either, because $[S(b(x)), x] = [N \setminus E(x), x]$.

The next corollary states that for all $x \in X$, and independently of the exit procedure $\Gamma(x)$, the exit produced by a non-panic equilibrium coincides with

$E(x)$. Hence, there is only one outcome of non-panic equilibria, $[N \setminus E(x), x]$. Moreover, this outcome is the unique outcome that is not Pareto dominated by any outcome of other equilibria.

Corollary 1 *Let $x \in X$ be given and let $b(x)$ be a non-panic equilibrium of any exit procedure $\Gamma(x)$. Then:*

(a) $[S(b(x)), x] = [N \setminus E(x), x]$.

(b)   *Let $b'(x)$ be an equilibrium of $\Gamma(x)$ such that $[S(b(x)), x] \neq [S(b'(x)), x]$.* Then, $[S(b(x)), x]$ Pareto dominates $[S(b'(x)), x]$.

Proof (a) It is an immediate consequence of Proposition 4 and the definition of panic equilibrium.

(b) Let $b(x)$ be a non-panic equilibrium of $\Gamma(x)$. By (a) the outcome generated by $b(x)$ is $[N \setminus E(x), x]$ and assume that the outcome generated by another equilibrium $b'(x)$ is different. By Proposition 4, $S(b'(x)) \subsetneq N \setminus E(x)$. Take $i \in N$. We distinguish among three different cases:

Case 1: $i \in E(x)$. Then, by the indifference condition (C2),

$$[N \setminus E(x), x] \, I_i \, [\varnothing, x] \, I_i \, [S(b'(x)), x].$$

Case 2: $i \in S(b'(x))$. Then, $[N \setminus E(x), x] \, P_i \, [S(b'(x)), x]$ because preferences are monotonic.

Case 3: $i \in (N \setminus E(x)) \setminus S(b'(x))$. Then, $[N \setminus E(x), x] \, P_i \, [\varnothing, x]$ because $[N \setminus E(x), x]$ is the outcome induced by the equilibrium $b(x)$ and, by playing $b_i^e(x)$, member i can force an outcome $[T, x] \, I_i \, [\varnothing, x]$. By the indifference condition (C2),

$$[N \setminus E(x), x] \, P_i \, [S(b'(x)), x]$$

because $i \notin S(b'(x))$.
    Since $(N \setminus E(x)) \setminus S(b'(x)) \neq \varnothing$ we conclude that $[N \setminus E(x), x]$ Pareto dominates $[S(b'(x)), x]$.
    Assume that in the exit procedure $\Gamma(x)$ members always play a *SPNE* strategy $b(x)$ with the property that $S(b(x)) = N \setminus E(x)$. Then, computing the *SPNE* of the two stage game $\Upsilon = ((M, v), \{\Gamma(x)\}_{x \in X})$ is the same as computing the *NE* of the normal form game $\Delta = (N, M, R, o)$ where o is the outcome function defined as follows: for each $m \in M$,

$$o(m) = [N \setminus E(v(m)), v(m)].$$

For the case of choosing new members we characterize in Berga, Bergantiños, Massó, and Neme (2003) voting by quota n as the unique social choice function $f : \widetilde{\mathcal{R}} \to 2^N \times X$ (on the domain of candidate separable preferences $\widetilde{\mathcal{R}}$),[3] satisfy-

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3   See Subsection 4.3 for a definition of candidate separable preference profiles.

ing strategy-proofness, voter's sovereignty on *K*, and stability. The stability property has two components: internal stability and external stability. Internal stability says that members who remain in the society do not want to exit, whereas external stability says that members who leave the society do not want to rejoin it. We can translate both concepts in terms of our game Δ as follows.

INTERNAL STABILITY: A strategy profile $m \in M$ satisfies *internal stability* if $i \in N \setminus E(v(m))$ implies $[N \setminus E(v(m)), v(m)]\ P_i\ [\varnothing, v(m)]$.

EXTERNAL STABILITY: A strategy profile $m \in M$ satisfies *external stability* if $i \notin N \setminus E(v(m))$ implies $[\varnothing, v(m)]\ P_i\ [N \setminus E(v(m)) \cup \{i\}, v(m)]$.

Although the internal stability of a *NE* of Δ follows immediately from the definition of *E*, the external stability of a *NE* does not hold so trivially. The reason is that if member *i*, who exits in equilibrium, changes his strategy and chooses to stay, for instance by playing b_i^s, this could affect the choice of other members and hence, the final outcome could potentially be different from $[N \setminus E(v(m)) \cup \{i\}, v(m)]$. However, Proposition 5 states that this is never the case; that is, all *NE* of Δ = (*N, M, R, o*) satisfy internal and external stability, whenever the preference profile *R* is monotonic.

Proposition 5 *Let* $m \in M$ *be a NE of* Δ = (*N, M, R, o*), *where R is a monotonic preference profile. Then, m satisfies internal and external stability.*

Proof Assume *m* is a *NE* of Δ and $i \in N \setminus E(v(m))$. Since $i \notin E^{t_{v(m)}+1}(v(m))$, $[N \setminus E(v(m)), v(m)]\ P_i\ [\varnothing, v(m)]$, which means that *m* satisfies internal stability.

Assume *m* is a *NE* of Δ and $i \notin N \setminus E(v(m))$. Therefore, there exists *t* such that $i \in E^t(v(m))$. Hence,

$$[\varnothing, v(m)]\ P_i\ \left[N \setminus \left(\bigcup_{t'=1}^{t-1} E^{t'}(v(m)) \right), v(m) \right].$$

Since $N \setminus E(v(m)) \subset N \setminus \left(\bigcup_{t'=1}^{t-1} E^{t'}(v(m)) \right)$ and R_i is monotonic,

$$\left[N \setminus \left(\bigcup_{t'=1}^{t-1} E^{t'}(v(m)) \right), v(m) \right]\ P_i\ [(N \setminus E(v(m))) \cup \{i\}, v(m)].$$

Therefore, by transitivity of P_i, $[\varnothing, v(m)]\ P_i\ [(N \setminus E(v(m))) \cup \{i\}, v(m)]$, which means that *m* satisfies external stability.

We have defined the exit set *E* when preference profiles are monotonic. However, we could define it for general preference profiles satisfying conditions (C1), (C2), and (C3). And thus, the definition of the game Δ would still be meaningful. But then, a *NE* of Δ still satisfies internal stability but not necessarily external stability. Example 6 exhibits an instance of such possibility.

Example 6 Consider again Example 2 except that now the preference profile R is additively representable by the following table

	u_1	u_2	u_3
1	1	–8	1
2	1	5	–10
3	1	12	15
y	100	–7	–8

Notice that R_2 and R_3 are not monotonic ($[\{2, 3\}, \varnothing] P_2 [N, \varnothing]$ and $[\{1, 3\}, \varnothing] P_3 [N, \varnothing]$). It is immediate to see that $E(\varnothing) = \varnothing$. Moreover, $E^1(y) = \{3\}$, $E^2(y) = \{2\}$, and $E^3(y) = \varnothing$. Hence, $E(y) = \{2, 3\}$.

Assume that the voting procedure is voting by quota 1 and that $m \in \{\varnothing, y\}^3$ is such that $vc^1(m) = \varnothing$. Then, $m_i = \varnothing$ for all $i \in N$. If member 1 votes for y instead of voting for $\varnothing$, $vc^1(y, m_{-1}) = y$ and hence,

$$[N \setminus E(v(y, m_{-1})), v(y, m_{-1})] = [\{1\}, \{y\}] P_1 [N, \varnothing] = [N \setminus E(v(m)), v(m)],$$

which means that m is not a *NE* of Δ.

It is easy to see that $[\{1\}, \{y\}]$ is the final society generated by the *NE* strategy $\bar{m} = (y, \varnothing, \varnothing)$. Moreover, it is the unique final society that can be generated by a *NE* of Δ. But $\bar{m}$ does not satisfy external stability because

$$[\{1, 3\}, \{y\}] P_3 [\varnothing, \{y\}].$$

4.1. Simultaneous exit

In this subsection we study the relationship between the exit set after x is chosen, $E(x)$, with plausible outcomes of the extensive form game when exit is simultaneous and preference profiles are monotonic. In particular, we show that $E(x)$ coincides with the outcome of applying the process of iterative elimination of dominated strategies (*IEDS*).[4]

Given $x \in X$ and $i \in N$ we say that $b_i''(x)$ is *dominated* if there exists $b_i'(x)$ satisfying two conditions. First, for all $b(x) \in B(x)$,

$$[S(b_i'(x), b_{-i}(x)), x] R_i [S(b_i''(x), b_{-i}(x)), x].$$

Second, there exists $b^*(x) \in B(x)$ such that

$$[S(b_i'(x), b_{-i}^*(x)), x] P_i [S(b_i''(x), b_{-i}^*(x)), x].$$

For all $x \in X$ and $i \in N$ we can define the set $B_i^{nd}(x)$ as the set of strategies of member i which survive the process of *IEDS*.

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[4]   For a formal definition of the process of *IEDS* see, for instance, van Damme (1991).

The next proposition states that, given $x \in X$, the strategy s of member $i \in E(x)$ in the simultaneous game $\Gamma(x)$ does not survive the process of *IEDS*.

Proposition 6 *For all* $x \in X$, $E(x) = \{i \in N \mid B_i^{nd}(x) = \{e\}\}$.

Proof Remember that $B_i(x) = \{e, s\}$ for all $i \in N$ and $E(x) = \displaystyle\bigcup_{t=1}^{t_x} E^t(x)$.

We first prove that $B_i^{nd}(x) = \{e\}$ for all $i \in E^1(x)$. Given $b(x) \in B(x)$ and $i \in E^1(x)$ we define $b'(x) = (e, b_{-i}(x))$ and $b''(x) = (s, b_{-i}(x))$. It is easy to see that $S(b''(x)) = S(b'(x)) \cup \{i\}$. Since preferences are monotonic, $i \in E^1(x)$, and $i \notin S(b'(x))$, we obtain

$$[S(b'(x)), x] \ I_i \ [\varnothing, x] \ P_i \ [N, x] \ R_i \ [S(b''(x)), x].$$

Then, s is dominated and hence, $B_i^{nd}(x) = \{e\}$. Moreover, by the definition of $E^1(x)$, if strategy s of member $i \in N$ is eliminated in the first step of the process of *IEDS*, $i \in E^1(x)$.

We now prove that $B_i^{nd}(x) = \{e\}$ for all $i \in E^2(x)$. Given $i \in E^2(x)$ and $b(x) \in B(x)$ such that $b_j(x) = e$ for all $j \in E^1(x)$ we define $b'(x) = (e, b_{-i}(x))$ and $b''(x) = (s, b_{-i}(x))$. It is easy to see that $S(b''(x)) = S(b'(x)) \cup \{i\}$ and $E^1(x) \cap S(b'(x)) = \varnothing$. Since preferences are monotonic, $i \in E^2(x)$, and $i \notin S(b'(x))$, we obtain

$$[S(b'(x)), x] \ I_i \ [\varnothing, x] \ P_i \ [N \setminus E^1(x), x] \ R_i \ [S(b''(x)), x].$$

Then, s is dominated and hence, $B_i^{nd}(x) = \{e\}$. Moreover, by the definition of $E^2(x)$, if strategy s of member $i \in N$ is eliminated in the second step of the process of *IEDS*, $i \in E^2(x)$.

Repeating this argument, we conclude that $B_i^{nd}(x) = \{e\}$ for all $i \in E^t(x)$ and $t = 3, ..., t_x$. Moreover, by definition of $E^t(x)$, if strategy s of any member $i \in N$ is eliminated in the tth step of the process of *IEDS*, $i \in E^t(x)$. Then, $E(x) \subset \{i \in N \mid B_i^{nd}(x) = \{e\}\}$.

We only need to prove that if $i \notin E(x)$ then, $s \in B_i^{nd}(x)$. Since preferences are monotonic, it is enough to prove that in step $t_x + 1$ of the process of *IEDS* no strategy s can be eliminated. Take $i \notin E(x)$. If strategy e of member i was eliminated then s can not obviously be eliminated in step $t_x + 1$. Assume that strategy e was not eliminated. Consider $b(x)$ such that $b_j(x) = e$ for all $j \in E(x)$ and $b_j(x) = s$ for all $j \in N \setminus (E(x) \cup \{i\})$. Notice that all these strategies are available for members in step $t_x + 1$ of the process of *IEDS*. Since $i \notin E^{t_x+1}(x)$,

$$[S(s, b_{-i}(x)), x] = [N \setminus E(x), x] \ P_i \ [\varnothing, x] \ I_i \ [S(e, b_{-i}(x)), x].$$

Hence, s can not be eliminated.

4.2. Sequential exit

In this subsection we prove that whenever the exit procedure is sequential then the exit set after x, $E(x)$, coincides with the set of members that leave the

society in the *SPNE* of $\Gamma^\sigma(x)$, for all orderings $\sigma \in \Sigma$. Therefore, for all $x \in X$, the *SPNE* outcome of $\Gamma^\sigma(x)$ coincides with $[N \setminus E(x), x]$.

Proposition 7 *Let $x \in X$ and $\sigma \in \Sigma$ be given, and let $b(x)$ be the SPNE of $\Gamma^\sigma(x)$. Then, $E(x) = E(b(x))$.*

Proof To simplify notation we assume, without loss of generality, that $\sigma(i) = i$ for all $i \in N$.

We define the following sets. $S^1 = \varnothing$. Assume that we have defined S^j for all $j < i$. We now define S^i as

$$S^i = \begin{cases} S^{i-1} & \text{if } (b_{i-1}(x))(S^{i-1}) = e \\ S^{i-1} \cup \{i-1\} & \text{if } (b_{i-1}(x))(S^{i-1}) = s. \end{cases}$$

We must prove that $(b_i(x))(S^i) = e$ when $i \in E(x)$ and $(b_i(x))(S^i) = s$ when $i \notin E(x)$.

By Proposition 4 we know that $E(x) \subset E(b(x))$. Then, $(b_i(x))(S^i) = e$ when $i \in E(x)$.

Assume that $N \setminus E(x) = \{i_1, ..., i_l\}$ and $i_j < i_{j+1}$ for all $j = 1, ..., l-1$. We now prove that given $T^l = \{i_1, ..., i_{l-1}\}$ we have that $(b_{i_l}(x))(T^l) = s$. Since $\{i_l + 1, ..., n\} \subset E(x)$, using arguments similar to those used in the proof of Proposition 4 we can show that, independently of the action chosen by i_l, s or e, members of $\{i_l + 1, ..., n\}$ will play e in any *SPNE*. Then, if i_l chooses s, then the final society is $[T^l \cup \{i_l\}, x]$, whereas if i_l had chosen e, then the final society would be $[T^l, x]$. Since $i_l \notin E^{t_x+1}(x)$ we know that $[N \setminus E(x), x] \ P_{i_l} [\varnothing, x]$. Then,

$$[T^l \cup \{i_l\}, x] = [N \setminus E(x), x] \ P_{i_l} \ [\varnothing, x] I_{i_l} [T^l, x],$$

where $[\varnothing, x] \ I_{i_l} \ [T^l, x]$ comes from the indifference condition (C2). This means that $(b_{i_l}(x)) (T^l) = s$ because $b(x)$ is the *SPNE* of $\Gamma^\sigma(x)$. We now prove that given $T^{l-1} = \{i_1, ..., i_{l-2}\}$ we have that $(b_{i_{l-1}}(x))(T^{l-1}) = s$. Using arguments similar to those used in the proof of Proposition 4 we can show that, independently of the action chosen by i_{l-1}, s or e, members of $\{i_{l-1} + 1, ..., n\} \cap E(x)$ will play e in any *SPNE*. Then, if i_{l-1} chooses s, the information set T^l of i_l will be reached. But, since we have already proven that member i_l will choose s in T^l, the final society will be $[N \setminus E(x), x]$. If member i_{l-1} chooses e, the final society will be $[T^*, x]$ where $T^* = T^{l-1}$ or $T^* = T^{l-1} \cup \{i_l\}$. In any case, $[\varnothing, x] I_{i_{l-1}} [T^*, x]$. Since $i_{l-1} \notin E^{t_x+1}(x)$ we know that $[N \setminus E(x), x] P_{i_{l-1}} [\varnothing, x]$. Then,

$$[N \setminus E(x), x] P_{i_{l-1}} [T^*, x].$$

Hence, $(b_{i_{l-1}}(x))(T^{i_{l-1}}) = s$ because $b(x)$ is the *SPNE* of $\Gamma^\sigma(x)$.

Repeating this same argument several times we obtain that $(b_{i_j}(x))(T^j) = s$ for all $j = 1, ..., l$. Now, the result follows immediately because $T^j = S^{i_j}$ whenever $i_j \in N \setminus E(x)$.

4.3. An application

Consider again the problem where the initial society has to choose new members of the society from a given set K of candidates.[5] As in Barberà, Sonnenschein, and Zhou (1991) we will assume that members of the initial society order final societies according to whether or not a candidate is good or bad. Let $R_i \in \mathcal{R}_i$ be a preference relation of member $i \in N$ and let $y \in K$. We say that candidate y is *good* for member i according to R_i whenever $[N, \{y\}]\, P_i\, [N, \varnothing]$; otherwise, we say that candidate y is *bad* for member i according to R_i. Denote by $G_K(R_i)$ and $B_K(R_i)$ the set of good and bad candidates for i according to R_i, respectively. Given a preference profile $R = (R_1, ..., R_n) \in \mathcal{R}$, let

$$G(R) = \underset{i \in N}{\mathrm{I}}\, G_K(R_i) \text{ the set of unanimously good candidates and } B(R) = \underset{i \in N}{\mathrm{I}}\, B_K(R_i)$$

the set of unanimously bad candidates.

A preference relation $R_i \in \mathcal{R}_i$ is *candidate separable* if for all $T \subset N$ such that $i \in T$, all $S \subset K$, and $y \in K \setminus S$,

$$[T, S \cup \{y\}]\, P_i\, [T, S] \text{ if and only if } y \in G_K(R_i).$$

Let $S_i \subset \mathcal{R}_i$ be the set of monotonic and candidate separable preference relations of member i and let S denote the Cartesian product $S_1 \times \cdots \times S_n$.

We now study the *NE* of Δ, when members leaving the society is the exit set after x and the voting rule is a voting by committees vc; that is, the game is $\Delta = (N, M, R, o)$, where $M_i = 2^K$ for all $i \in N$ and $o(m) = [N \setminus E(vc(m)), v(m)]$. Remember that this is equivalent to study the *SPNE* of $\Upsilon = \left((2^K)^N, vc\right), \{\Gamma^{\sigma}(x)\}_{x \in 2^K}$ with sequential exit.

Proposition 8 *Let $vc : (2^K)^N \to 2^K$ be a voting by committees without dummies and let $R \in S$ be a monotonic and candidate separable preference profile. Then, the strategy m_i of voting for a common bad ($m_i \cap B(R) \neq \varnothing$) and the strategy m'_i of not voting for a common good ($G(R) \to (K \setminus m'_i) \neq \varnothing$) are dominated strategies in Δ.*

Proof We will only show that to vote for a common bad is a dominated strategy. The proof that to not vote for a common good is also a dominated strategy is similar and left to the reader. Assume vc is a voting by committees without dummies and $R \in S$ is a monotonic and candidate separable preference profile. Consider member $i \in N$ and a strategy $m_i \in 2^K$ with the property that $y \in m_i \cap B(R)$. We will show that the strategy $m'_i = m_i \setminus \{y\}$ dominates m_i. Fix $m_{-i} \in M_{-i}$ and consider the two subsets of candidates $vc(m)$ and $vc(m) \setminus \{y\}$. We first prove the following claim:

CLAIM: $E(vc(m) \setminus \{y\}) \subset E(vc(m))$.

PROOF OF THE CLAIM: By definition, $E(vc(m) \setminus \{y\}) = \bigcup_{t=1}^{T'} E^t(vc(m) \setminus \{y\})$ and

$E(vc(m)) = \bigcup_{t=1}^{T} E^t(vc(m))$, where $T' = t_{vc(m) \setminus \{y\}}$ and $T = t_{vc(m)}$. We first show that

$E^1(vc(m) \setminus \{y\}) \subset E(vc(m))$. Assume $j \in E^1(vc(m) \setminus \{y\})$. Then,

(1) $\qquad\qquad\qquad [\varnothing, vc(m) \setminus \{y\}] \, P_j \, [N, vc(m) \setminus \{y\}].$

Since $y \in B_K(R_j)$ and R_j is candidate separable, $[N, vc(m) \setminus \{y\}] \, P_j \, [N, vc(m)]$. Therefore, by the indifference condition (C2), and the transitivity of R_j we conclude that

$$[\varnothing, vc(m)] \, P_j \, [N, vc(m)].$$

Thus, $j \in E^1(vc(m)) \subset E(vc(m))$. Assume now that $E^t(vc(m) \setminus \{y\}) \subset E(vc(m))$ for all $t = 1, ..., t_0 - 1$, where $2 \le t_0 \le T'$. We now prove that $E^{t_0}(vc(m) \setminus \{y\}) \subset E(vc(m))$. Suppose not. Then, there exists $j \in E^{t_0}(vc(m) \setminus \{y\}) \setminus E(vc(m))$. Since $j \in E^{t_0}(vc(m) \setminus \{y\})$,

$$[\varnothing, vc(m) \setminus \{y\}] \, P_j \, \left[N \setminus \left(\bigcup_{t=1}^{t_0-1} E^t(vc(m) \setminus \{y\}) \right), vc(m) \setminus \{y\} \right].$$

Then,

$$\left[N \setminus \left(\bigcup_{t=1}^{t_0-1} E^t(vc(m) \setminus \{y\}) \right), vc(m) \setminus \{y\} \right] P_j \, [N \setminus E(vc(m)), vc(m) \setminus \{y\}]$$

because preferences are monotonic and $\bigcup_{t=1}^{t_o-1} E^t(vc(m) \setminus \{y\}) \subset E(vc(m))$ by assumption. Since $y \in B_K(R_j)$ and R_j is candidate separable

$$[N \setminus E(vc(m)), vc(m) \setminus \{y\}] \, P_j \, [N \setminus E(vc(m)), vc(m)].$$

Moreover,

$$[N \setminus E(vc(m)), vc(m)] \, P_j \, [\varnothing, vc(m)]$$

because $j \notin E(vc(m))$. Hence, by the transitivity of R_j, $[\varnothing, vc(m) \setminus \{y\}] \, P_j \, [\varnothing, vc(m)]$, which contradicts the indifference condition (C2). Therefore, the Claim is proved.

We now compare the outcomes $o(m_i', m_{-i})$ and $o(m_i, m_{-i})$ in the three following mutually exclusive cases:

1. Suppose that $i \in E(vc(m) \setminus \{y\})$. By the above Claim, $i \in E(vc(m))$. Therefore, by the indifference condition (C2), $o(m_i', m_{-i}) \, I_i \, o(m_i, m_{-i})$.

2. Suppose that $i \notin E(vc(m) \setminus \{y\})$ and $i \in E(vc(m))$. Hence,

$$[N \setminus E(vc(m) \setminus \{y\}), vc(m) \setminus \{y\}] \, P_i \, [\varnothing, vc(m) \setminus \{y\}] \, I_i \, [\varnothing, vc(m)] \, .$$

Since $vc(m_i', m_{-i})$ is equal to either $vc(m)$ or $vc(m) \setminus \{y\}$,

$$
\begin{aligned}
o(m_i', m_{-i}) &= [N \setminus E(vc(m_i', m_{-i})), vc(m_i', m_{-i})] \\
&\quad R_i \, [N \setminus E(vc(m_i, m_{-i})), vc(m_i, m_{-i})] \\
&= o(m_i, m_{-i}).
\end{aligned}
$$

3. Suppose that $i \notin E(vc(m) \setminus \{y\})$ and $i \notin E(vc(m))$. Hence,

$$
\begin{aligned}
[N \setminus E(vc(m) \setminus \{y\}), vc(m) \setminus \{y\}] \ \ &P_i \, [N \setminus E(vc(m)), vc(m) \setminus \{y\}] \\
&P_i \, [N \setminus E(vc(m)), vc(m)],
\end{aligned}
$$

where the two strict preferences follow from the monotonicity (and the above Claim) and the candidate separability of R_i, respectively.

Since vc is without dummies we can find $I \in W_y^m$ such that $i \in I$. Take $m_j^* = \{y\}$ for all $j \in I \setminus \{i\}$, $m_j^* = \varnothing$ for all $j \in N \setminus I$, $m_i' = \varnothing$, and $m_i = \{y\}$. Then, $vc(m_i, m_{-i}^*) = \{y\}$ and $vc(m_i', m_{-i}^*) = \varnothing$, and hence, by the non-initial exit condition (C3), $i \notin E(vc(m_i, m_{-i}^*) \setminus \{y\}) = E(vc(m_i', m_{-i}^*)) = E(\varnothing) = \varnothing$.

If $i \in E(y)$ then,

$$o(m_i', m_{-i}^*) = [N, \varnothing] \, P_i \, [\varnothing, \varnothing] \, I_i \, [\varnothing, \{y\}] \, I_i \, [N \setminus E(y), \{y\}] = o(m_i, m_{-i}^*)$$

because of the indifference condition (C2) and the non-initial exit condition (C3).

If $i \notin E(y)$ then,

$$o(m_i, m_{-i}^*) = [N, \varnothing] \, P_i \, [N \setminus E(y), \{y\}] = o(m_i, m_{-i}^*)$$

because $y \in B_K(R_i)$ and preferences are monotonic and candidate separable.

In both cases, $o(m_i', m_{-i}^*) \, P_i \, o(m_i, m_{-i}^*)$. Therefore, $o(m_i', m_{-i}) \, R_i \, o(m_i, m_{-i})$ for all m_{-i} and there exists at least one $m_{-i}^* \in M_{-i}$ for which $o(m_i', m_{-i}^*) \, P_i \, o(m_i, m_{-i}^*)$. Thus, strategy m_i is dominated by strategy m_i'.

Remark 4 In Proposition 8 we assumed that the voting by committees vc had no dummies. Notice that if member i is a dummy for y, then to vote m_i and to vote $m_i \setminus \{y\}$ are equivalent strategies for member i because, independently of what the rest of members are voting, a vote of m_i or $m_i \setminus \{y\}$ leads to the same final outcome.

Adapting the concept of voter's sovereignty from Berga, Bergantiños, Massó, and Neme (2003) we say that a *NE* strategy $m \in M$ of Δ satisfies *voter's sovereignty* if $G(R) \subset vc(m) \subset N \setminus B(R)$. The next corollary is an immediate consequence of Proposition 8.

Corollary 2 *Let $\Delta = (N, (2^K)^N, R, o)$ be a normal form game associated with a voting by committees without dummies vc and with a monotonic and candidate separable preference profile $R \in S$. Then, all undominated NE of Δ satisfy voter's sovereignty.*

   We have established in Proposition 5 and Corollary 2 that all undominated *NE* of Δ satisfy stability (internal and external) and voter's sovereignty. The next example shows that, unfortunately, the set of undominated *NE* of Δ might be empty.

Example 7 Consider a society $N = \{1, 2, 3, 4\}$, whose members have to decide whether or not to admit as new members of the society candidates x and y. Suppose that the voting procedure $(\{\varnothing, x, y, \{x, y\}\}^N, vc^1)$ is voting by quota one and the preference profile R is representable by the following table (as in Example 2):

	u_1	u_2	u_3	u_4
1	100	1	1	1
2	1	100	1	1
3	1	100	1	1
4	100	1	1	2
x	2	−1	−5	−3
y	−1	2	−5	−3

   For member 1, $\{y\}$ is dominated by $\varnothing$ and $\{x, y\}$ is dominated by $\{x\}$. For member 2, $\{x\}$ is dominated by $\varnothing$ and $\{x, y\}$ is dominated by $\{y\}$. For members 3 and 4, $\{x\}$, $\{y\}$, and $\{x, y\}$ are dominated by $\varnothing$. Therefore, the undominated strategies are $\{x\}$ and $\varnothing$ for member 1; $\{y\}$ and $\varnothing$ for member 2; $\varnothing$ for member 3; and $\varnothing$ for member 4. In the next table we list all possible strategy profiles with undominated strategies and their corresponding final societies.

Voting	Final society
$(\varnothing, \varnothing, \varnothing, \varnothing)$	$[N, \varnothing]$
$(\varnothing, \{y\}, \varnothing, \varnothing)$	$[\{1, 2, 4\}, \{y\}]$
$(\{x\}, \varnothing, \varnothing, \varnothing)$	$[\{1, 2, 4\}, \{x\}]$
$(\{x\}, \{y\}, \varnothing, \varnothing)$	$[\{1, 2\}, \{x, y\}]$

   We now check that none of the four strategy profiles are *NE* of Δ.

1. $(\varnothing, \varnothing, \varnothing, \varnothing)$ is not an equilibrium because member 1 improves by voting x (1 prefers $[\{1, 2, 4\}, \{x\}]$ to $[N, \varnothing]$).

2. $(\emptyset, \{y\}, \emptyset, \emptyset)$ is not an equilibrium because member 2 improves by voting $\emptyset$ (2 prefers $[N, \emptyset]$ to $[\{1, 2, 4\}, \{y\}]$).

3. $(\{x\}, \emptyset, \emptyset, \emptyset)$ is not an equilibrium because member 2 improves by voting y (2 prefers $[\{1, 2\}, \{x, y\}]$ to $[\{1, 2, 4\}, \{x\}]$).

4. $(\{x\}, \{y\}, \emptyset, \emptyset)$ is not an equilibrium because member 1 improves by voting $\emptyset$ (1 prefers $[\{1, 2, 4\}, \{y\}]$ to $[\{1, 2\}, \{x, y\}]$).

Therefore, the set of undominated *NE* of Δ is empty. Moreover, it is easy to check that the set of Nash equilibria is equal to

$$\{m \in M \mid \#\{i \in N \mid x \in m_i\} \geq 2 \text{ and } \#\{i \in N \mid y \in m_i\} \geq 2\}.$$

Example 7 shows that in general it is not possible to find reasonable *NE*. This suggests the convenience of making additional assumptions on preference profiles. But this is outside the scope of this paper.

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